POLYNOMIALS, EXPONENTIAL AND LOGARITHMIC FUNCTION

Form 6

Vol 7

Part 1 – Division of polynomial

1.

$$\begin{array}{r} -2x^2+2x+1 \\ x+1\,\overline{)\,-2x^3 \qquad\quad +3x-1} \\ \underline{-2x^3-2x^2 \qquad\qquad} \\ 2x^2+3x \quad \\ \underline{2x^2+2x \quad} \\ x-1 \\ \underline{x+1} \\ -2 \end{array}$$

$\therefore$ Quotient $= -2x^2+2x+1$

Remainder $= -2$

2.

$$\begin{array}{r} x+2 \\ 3x^2+1\,\overline{)\,3x^3+6x^2+x-4} \\ \underline{3x^3 \qquad\quad +x \quad} \\ 6x^2 \qquad -4 \\ \underline{6x^2 \qquad +2} \\ -6 \end{array}$$

$\therefore$ Quotient $= x+2$

Remainder $= -6$

3.

$$\begin{array}{r|rrrr} -1 & 2 & -1 & 0 & -5 \\ & & -2 & 3 & -3 \\ \hline & 2 & -3 & 3 & -8 \end{array}$$

$\therefore$ Quotient $= 2x^2-3x+3$

Remainder $= -8$

4. $$\begin{array}{c|cccc} \frac{1}{3} & 3 & 5 & -8 & 6 \\ & & 1 & 2 & -2 \\ \hline & 3 & 6 & -6 & 4 \end{array}$$

$\therefore$ Quotient $= x^2 + 2x - 2$

Remainder $= 4$

Part 2 – Remainder/Factor theorem

1. Since $f(2) = 15$, we have $(2)^3 + k(2)^2 - 1 = 15$.
   Thus, we have $k = 2$

2. Note that $g(2) = 15$ and $g(-1) = -3$.
   Hence, we have $(2)^3 + (2)^2 + a(2) + b = 15$ and $(-1)^3 + (-1)^2 + a(-1) + b = -3$.
   So, we have $2a + b = 3$ and $a - b = 3$.
   Solving, we have $a = 2$ and $b = -1$.

3. Note that $h(\frac{3}{2}) = R$.

   Required remainder

   $= 1 - 2h(\frac{6}{4})$

   $= 1 - 2h(\frac{3}{2})$

   $= 1 - 2R$

4. Since $p(-3) = 0$, we have $(-3)^2 - a(-3) - b = 0$.
   So, we have $3a - b + 9 = 0$.
   Thus, we have $6a - 2b + 10 = 2(3a - b + 9) - 8 = -8$.

5. Since $f(-1) = 0$, we have $(-1)^{2k} + a(-1)^3 + b(-1) = 0$.
   Since $2k$ is even, we have $(-1)^{2k} = 1$ and hence $a + b = 1$.
   Required remainder
   $= f(1)$
   $= (1)^{2k} + a(1)^3 + b(1)$
   $= 1 + a + b$
   $= 2$

6. Since $g(-2)=0$, we have $(-2)^4+a(-2)^3-b(-2)^2+4=0$.
So, we have $2a+b=5$ and hence $b=5-2a$.
Required remainder
$=g(2)$
$=2^4+a(2)^3-b(2)^2+4$
$=20+8a-4b$
$=20+8a-4(5-2a)$
$=16a$

Part 3 – Factorization of polynomial

1. (a) Note that $f(3)=-84$ and $f(-2)=21$.
Hence, we have $2(3)^3+p(3)^2-4q(3)-15=-84$ and $2(-2)^3+p(-2)^2-4q(-2)-15=21$.
So, we have $3p-4q=-41$ and $p+2q=13$.
Solving, we have $p=-3$ and $q=8$.
(b) From (a), we have $f(x)=2x^3-3x^2-32x-15$.
$f(5)$
$=2(5)^3-3(5)^2-32(5)-15$
$=0$
(c) Since $f(5)=0$, $x-5$ is a factor of $f(x)$.
$f(x)=0$
$(x-5)(2x^2+7x+3)=0$
$(x-5)(x+3)(2x+1)=0$
$x=5$ or $x=-3$ or $x=-\dfrac{1}{2}$

2. (a) Note that $f(5)=26$ and $f(3)=0$.
Hence, we have $2(5)^3+a(5)^2+3(5)+b=26$ and $2(3)^3+a(3)^2+3(3)+b=0$.
So, we have $25a+b=-239$ and $9a+b=-63$.
Solving, we have $a=-11$ and $b=36$.
(b) From (a), we have $f(x)=2x^3-11x^2+3x+36$.
$f(x)=0$
$(x-3)(2x^2-5x-12)=0$
$(x-3)(x-4)(2x+3)=0$
$x=3$ or $x=4$ or $x=-\dfrac{3}{2}$.
Note that $-\dfrac{3}{2}$ is not an integer.
Thus, not all the roots of the equation $f(x)=0$ are integers.
The claim is disagreed.

3. (a) Note that $f(1) = 0$ and $g(1) = 0$.

Hence, we have $a(1)^3 - b(1)^2 - 1 + 2 = 0$ and $(1)^3 - (1)^2 - a(1) + 2b = 0$.

So, we have $a - b = -1$ and $a - 2b = 0$.

Solving, we have $a = -2$ and $b = -1$.

(b) $f(x) - g(x) = (-2x^3 + x^2 - x + 2) - (x^3 - x^2 + 2x - 2)$

$= -3x^3 + 2x^2 - 3x + 4$

Since $f(1) - g(1) = 0$, $x - 1$ is a factor of $f(x) - g(x)$.

$f(x) - g(x) = 0$

$(x-1)(-3x^2 - x - 4) = 0$

$x - 1 = 0$ or $-3x^2 - x - 4 = 0$

$(-1)^2 - 4(-3)(-4)$

$= -47$

< 0

So, the quadratic equation $-3x^2 - x - 4 = 0$ does not have real roots.

$\therefore\ x = 1$

4. (a) Note that $f(1) = 0$ and $f(-3) = -2k$.

Hence, we have $(1-3)^2(1+h) + k = 0$ and $(-3-3)^2(-3+h) + k = -2k$.

So, we have $4h + k = -4$ and $12h + k = 36$.

Solving, we have $h = 5$ and $k = -24$.

(b) From (a), we have $f(x) = (x-3)^2(x+5) - 24$.

$f(x) = 0$

$x^3 - x^2 - 21x + 21 = 0$

$(x-1)(x^2 - 21) = 0$

$x = 1$ or $x = \sqrt{21}$ or $x = -\sqrt{21}$

Note that $\sqrt{21}$ and $-\sqrt{21}$ are not rational numbers, while 1 is a rational root of the equation $f(x) = 0$.

Thus, the equation $f(x) = 0$ has 1 rational root.

Part 4 – Special Type

1. Let $f(x)=(x-2)q(x)$, where $q(x)$ is a polynomial.
   Then, we have $f(x+5)=(x+5-2)q(x+5)$.
   So, we have $f(x+5)=(x+3)q(x+5)$.
   Thus, $x+3$ is a factor of $f(x+5)$.

2. Let $f(x-3)=(x+5)q(x)$, where $q(x)$ is a polynomial.
   Then, we have $f(2x+1)=(2x+4+5)q(2x+4)$.
   So, we have $f(2x+1)=(2x+9)q(2x+4)$.
   Thus, $2x+9$ is a factor of $f(2x+1)$.

3. Note that $x^2-1=(x-1)(x+1)$.
   Then, we have $f(1)=2(1)-3=-1$ and $f(-1)=2(-1)-3=-5$.
   Hence, we have $(1)^3-2(1)^2+a(1)+b=-1$ and $(-1)^3-2(-1)^2+a(-1)+b=-5$.
   So, we have $a+b=0$ and $a-b=2$.
   Solving, we have $a=1$ and $b=-1$.

4. (a) Let $ax+b$ be the required quotient, where a and b are constants.
   Then, we have $f(x)=(ax+b)(3x^2-4x+17)$.
   Note that $f(1)=80$ and $f(-3)=56$.
   Hence, we have $(a(1)+b)(3(1)^2-4(1)+17)=80$ and $(a(-3)+b)(3(-3)^2-4(-3)+17)=56$.
   So, we have $a+b=5$ and $3a-b=-1$.
   Solving, we have $a=1$ and $b=4$.
   Thus, the required quotient is $x+4$.

   (b) $f(x)=0$
   $(x+4)(3x^2-4x+17)=0$
   $(-4)^2-4(3)(17)$
   $=-188$
   <0
   So, the quadratic equation $3x^2-4x+17=0$ does not have real roots.
   Note that -4 is a rational root of the equation $f(x)=0$.
   Thus, the equation $f(x)=0$ has 1 rational root.

5. (a) Let $ax+b$ be the required remainder, where a and b are constants.

Then, we have $f(x)=(x^2-2x-3)q(x)+ax+b$, where $q(x)$ is a polynomial.

Note that $x^2-2x-3=(x+1)(x-3)$.

Hence, we have $f(-1)=-24$ and $f(3)=-120$.

So, we have $-a+b=-24$ and $3a+b=-120$.

Solving, we have $a=-24$ and $b=-48$.

Thus, the required remainder is $-24x-48$.

(b) Let $q(x)=mx+n$, where m and n are constants.

Then, we have $f(x)=(x^2-2x-3)(mx+n)-24x-48$.

Note that $x^2-2x-15=(x+3)(x-5)$.

Since $f(x)$ is divisible by $x^2-2x-15$, we have $f(-3)=0$ and $f(5)=0$.

Hence, we have $((-3)^2-2(-3)-3)(m(-3)+n)-24(-3)-48=0$ and

$((5)^2-2(5)-3)(m(5)+n)-24(5)-48=0$.

So, we have $3m-n=2$ and $5m+n=14$.

Solving, we have $m=2$ and $n=4$.

$f(x)=0$

$(x^2-2x-3)(2x+4)-24x-48=0$

$2(x^2-2x-3)(x+2)-24(x+2)=0$

$2(x+2)(x^2-2x-15)=0$

$2(x+2)(x+3)(x-5)=0$

$x=-2$ or $x=-3$ or $x=5$

6. (a) Let $ax+b$ be the required quotient.

Then, we have $f(x)=(ax+b)(x^2-8)-7$.

Hence, we have

$$\begin{aligned} f(x) &= (ax+b)(x^2+2-10)-7 \\ &= (ax+b)(x^2+2)-10ax-10b-7 \end{aligned}$$

Note that the remainder when $f(x)$ is divided by x^2+2 is $-20x-17$.

So, we have $-10a=-20$ and $-10b-7=-17$.

Thus, we have $a=2$ and $b=1$.

(b) $f(x)$

$=(2x+1)(x^2-8)-7$

$=2x^3+x^2-16x-15$

$=(x-3)(2x^2+7x+5)$

$=(x-3)(x+1)(2x+5)$

Therefore, the roots of the equation $f(x)=0$ are 3, -1 and $-\dfrac{5}{2}$.

All the roots of the equation $f(x)=0$ are rational numbers.

The claim is agreed.

7. Let $x^3-2x^2-6x+27=(ax+b)(x^2-5x+k)$, where a and b are constants.

By comparing the coefficient of x^3, we have $a=1$.

Note that the coefficient of x^2 and the constant term in the expansion of $(x+b)(x^2-5x+k)$ are $b-5$ and bk respectively.

So, we have $b-5=-2$ and $bk=27$.

Solving, we have $b=3$ and $k=9$.

8. Let $\text{f}(x)=(mx+n)(2x^2-bx-1)$, where a and b are constants.

By comparing the coefficient of x^3, we have $2m=8$ and hence $m=4$.

Note that the coefficient of x and the constant term in the expansion of $(4x+n)(2x^2-bx-1)$ are $-bn-4$ and $-n$ respectively.

So, we have $-bn-4=5$ and $-n=b$.

Hence, we have $b^2-4=5$.

Thus, we have $b=\pm 3$.

For $b=3$, we have $\text{f}(x)=(4x-3)(2x^2-3x-1)$.

Required remainder $=\text{f}(-1)=(4(-1)-3)(2(-1)^2-3(-1)-1)=-28$

For $b=-3$, we have $\text{f}(x)=(4x+3)(2x^2+3x-1)$.

Required remainder $=\text{f}(-1)=(4(-1)+3)(2(-1)^2+3(-1)-1)=2$

9. Let $4x^3+23x^2-35x-87=(ax+b)(x^2+8x+k)-3x+3$, where a and b are constants.

By comparing the coefficient of x^3, we have $a=4$.

Note that the coefficient of x^2 and the constant term in the expansion of $(4x+b)(x^2+8x+k)-3x+3$ are $b+32$ and $bk+3$ respectively.

So, we have $b+32=23$ and $bk+3=-87$.

Solving, we have $b=-9$ and $k=10$.

10. (a) Required remainder

$=\text{f}(1)$

$=(1)^{1001}-1$

$=0$

(b) Let $\text{f}(x)=(x-1)\text{q}(x)$, where $\text{q}(x)$ is a polynomial.

Then, we have $\text{f}(7)=(7-1)\text{q}(7)$.

Hence, we have $(7)^{1001}-7=6\text{q}(7)$.

So, we have $7^{1001}=6\text{q}(7)+7=6(\text{q}(7)+1)+1$.

Thus, the required remainder is 1.

11. (a) By comparing the coefficient of x^4, we have $l = 4$.

Note that the constant term in the expansion of $(4x^2 - 4x + m)(x^2 + nx + 2)$ is $2m$.

So, we have $2m = 2$ and hence $m = 1$.

Note that $f(-3) = 98$.

Hence, we have $(4(-3)^2 - 4(-3) + 1)((-3)^2 + n(-3) + 2) = 98$.

Thus, we have $n = 3$.

(b) (i) Note that $x^2 + 3x + 2 = (x+1)(x+2)$.

Hence, $x + 2$ is a factor of $f(x)$.

Therefore, we have $f(-2) = 0$.

Since the remainder when $g(x)$ is divided by $2x + 4$ is 0, we have $g(-2) = 0$.

Thus, we have $h(-2) = f(-2) - g(-2) = 0$.

It follows that $x + 2$ is a factor of $h(x)$.

(ii) Let $g(x) = (px + q)(2x + 4)$, where p and q are constants.

Then, we have $h(x) = (4x^2 - 4x + 1)(x^2 + 3x + 2) - (px + q)(2x + 4)$.

Note that $x^2 + x = x(x+1)$.

Hence, we have $h(0) = -10$ and $h(-1) = 0$.

So, we have $2 - 4q = -10$ and $-p + q = 0$.

Solving, we have $p = 3$ and $q = 3$.

$$h(x) = 0$$
$$(4x^2 - 4x + 1)(x^2 + 3x + 2) - (3x + 3)(2x + 4) = 0$$
$$(4x^2 - 4x + 1)(x + 1)(x + 2) - 6(x + 1)(x + 2) = 0$$
$$(x + 1)(x + 2)(4x^2 - 4x - 5) = 0$$

$x = -1$ or $x = -2$ or $4x^2 - 4x - 5 = 0$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(-5)}}{2(4)}$$

$$x = \frac{1 \pm \sqrt{6}}{2}$$

Note that $\frac{1 \pm \sqrt{6}}{2}$ are two irrational numbers.

The claim is agreed.

12. (a) Let $f(x) = (-x^2 + 4)q(x) + kx + 12$, where $q(x)$ is a polynomial.

Since $f(2) = 0$, we have $(-(2)^2 + 4)q(2) + k(2) + 12 = 0$.

Thus, we have $k = -6$.

(b) Let $f(x) = (x-2)(x+4)(ax+b)$, where a and b are constants.

Since $f(0) = -8$, we have $(-2)(4)(b) = -8$.

Solving, we have $b = 1$.

Note that $f(-2) = (-(-2)^2 + 4)q(-2) - 6(-2) + 12 = 24$.

Therefore, we have $((-2)-2)((-2)+4)(a(-2)+1) = 24$.

Solving, we have $a = 2$.

Hence, we have $f(x) = (x-2)(x+4)(2x+1)$.

The roots of the equation $f(x) = 0$ are 2, -4 and $-\frac{1}{2}$.

Note that $-\frac{1}{2}$ is not an integer.

Not all the roots of the equation $f(x) = 0$ are integers.

Thus, the claim is incorrect.

13. (a) Let the required quotient be $lx^2 + mx + n$, where l, m and n are constants.

Then, we have $f(x) = (lx^2 + mx + n)(3x^2 + bx - 3)$.

By comparing the coefficient of x^4, we have $3l = 3$ and hence $l = 1$.

Note that the coefficient of x^2 and the constant term in the expansion of $(x^2 + mx + n)(3x^2 + bx - 3)$ are $bn + 3n - 3$ and $-3n$ respectively.

So, we have $bm + 3n - 3 = b$ and $-3n = -3$.

Solving, we have $m = 1$ and $n = 1$.

Thus, the required quotient is $x^2 + x + 1$.

(b) From (a), we have $f(x) = (3x^2 + bx - 3)(x^2 + x + 1)$.

$f(x) = 0$

$(3x^2 + bx - 3)(x^2 + x + 1) = 0$

$3x^2 + bx - 3 = 0$ or $x^2 + x + 1 = 0$

For $3x^2 + bx - 3 = 0$, $\Delta = b^2 - 4(3)(-3) = b^2 + 36 > 0$.

So, the quadratic equation $3x^2 + bx - 3 = 0$ has two distinct real roots.

For $x^2 + x + 1 = 0$, $\Delta = 1^2 - 4(1)(1) = -3 < 0$.

So, the quadratic equation $x^2 + x - 1 = 0$ has no real roots.

Hence, the equation $f(x) = 0$ must have exactly two distinct real roots.

The claim is agreed.