COORDINATE GEOMETRY(III)

Form 6
Vol 6

Part 4 – Shortest/farthest distance

1. Denote the centre of C by G.

Note that the coordinates of G are $(8,5)$.

Let $Q\left(q,-\dfrac{13q+115}{10}\right)$ be the point lying on L such that GQ is perpendicular to L.

Slope of $GQ=(-1)\times\left(-\dfrac{10}{13}\right)=\dfrac{10}{13}$

So, we have

$$\frac{5+\dfrac{13q+115}{10}}{q-8}=-\frac{10}{13}$$

$q=-5$

$\therefore\ Q(-5,-5)$

Note that the distance from P to L attains its minimum as G, P and Q are collinear.

The radius of $C=\sqrt{8^2+5^2+\left(\dfrac{64}{2}\right)}=11$

The distance between G and $Q=\sqrt{(-5-8)^2+(-5-5)^2}=\sqrt{269}$

Therefore, the required distance is $\sqrt{269}-11$.

<u>Alternatively,</u>

The coordinates of the centre of C are $(8,5)$.

The radius of $C=\sqrt{8^2+5^2+\left(\dfrac{64}{2}\right)}=11$

By the shortest distance formula, we have

Shortest distance from the centre of C to L

$$=\left|\frac{13(8)+10(5)+115}{\sqrt{13^2+10^2}}\right|$$

$=\sqrt{269}$

Therefore, the required distance is $\sqrt{269}-11$.

2. (a) Slope of $PQ = \dfrac{-18+8}{12-14} = 5$

Slope of $QR = \dfrac{-16+18}{2-12} = -\dfrac{1}{5}$

$\therefore$ Slope of $PQ \times$ Slope of $QR = 5 \times \left(-\dfrac{1}{5}\right) = -1$

$\therefore PQ \perp QR$

Since $\angle PQR = 90°$, G lies on PR.

Thus, the coordinates of $G = \left(\dfrac{14+2}{2}, \dfrac{-8-16}{2}\right) = (8, -12)$

(b) (i) Note that RA is perpendicular to L.

Slope of $RA = (-1) \times \left(-\dfrac{2}{3}\right) = \dfrac{2}{3}$

Let the coordinates of A be $\left(a, -\dfrac{3a+52}{2}\right)$.

$$\frac{-12+\dfrac{3a+52}{2}}{8-a} = \frac{2}{3}$$

$a = -4$

So, the coordinates of A are $(-4, -20)$.

The distance between A and R

$= \sqrt{(-4-2)^2 + (-20+16)^2}$

$= 2\sqrt{13}$

(ii) Slope of $PA = \dfrac{-8+12}{14-8} = \dfrac{2}{3}$

$\therefore$ Slope of PA = Slope of RA

So, A, R and P are collinear.

Note that G lies on PR.

Thus, A, R, G and P are collinear.

(iii) The radius of $C = \sqrt{(8-14)^2 + (-12+8)^2} = 2\sqrt{13}$

The distance between A and $G = 2\sqrt{13} + 2\sqrt{13} = 4\sqrt{13}$

The distance between P and $G = 2\sqrt{13}$

The required ratio

$= AG : PG$

$= 4\sqrt{13} : 2\sqrt{13}$

$= 2 : 1$

Part 5A – 4 centre questions

1.

	Centre	Triangle	Inside triangle	On triangle	Outside triangle
(a)	Incentre	Obtuse-angled triangle	✓		
(b)	Circumcentre	Right-angled triangle		✓	
(c)	Orthocentre	Acute-angled triangle	✓		
(d)	Centroid	Obtuse-angled triangle	✓		
(e)	Orthocentre	Right-angled triangle		✓	
(f)	Circumcentre	Obtuse-angled triangle			✓
(g)	Centroid	Acute-angled triangle	✓		
(h)	Orthocentre	Obtuse-angled triangle			✓

2. (a) True
   (b) True
   (c) True
   (d) True
   (e) True
   (f) True

3. Note that OA is vertical.
The horizontal line $y=1$ passes through the orthocentre of ΔOAB.
Let the coordinates of the orthocentre of ΔOAB be $H(h,1)$.

Slope of $AH = \dfrac{3-1}{0-h} = -\dfrac{2}{h}$

Slope of $OB = \dfrac{1-0}{4-0} = \dfrac{1}{4}$

Note that the line passing through A and H is perpendicular to OB.

$-\dfrac{2}{h} \times \dfrac{1}{4} = -1$

$h = \dfrac{1}{2}$

Thus, the coordinates of the orthocentre of ΔOAB are $\left(\dfrac{1}{2},1\right)$.

4. (a) The coordinates of C are $(4,-5)$.

(b) Note that AC is vertical.

The horizontal line $y=7$ passes through K.

Let the coordinates of K be $(k,7)$.

Slope of $BC=\dfrac{-5-7}{4-16}=1$

Note that the line passing through A and K is perpendicular to BC.

$\dfrac{7-9}{k-4}=-1$

$k=6$

Thus, the coordinates of K are $(6,7)$.

5. (a) $\mathrm{f}(x)=\dfrac{1}{4}(x^2-16x)+10$

$=\dfrac{1}{4}(x^2-16x+8^2-8^2)+10$

$=\dfrac{1}{4}(x-8)^2-6$

Thus, the coordinates of A are $(8,-6)$.

(b) The coordinates of B are $(8,-6+k)$.

(c) The coordinates of C are $(0,10)$.

Note that AB is vertical.

The horizontal line $y=10$ passes through P.

Let the coordinates of P be $(h,10)$.

Note that the line passing through A and P is perpendicular to BC.

$\dfrac{10+6}{h-8}\times\dfrac{10+6-k}{0-8}=-1$

$h=40-2k$

Putting $x=40-2k,\ y=10$ in $y=\mathrm{f}(x)+k$, we have

$10=\dfrac{1}{4}(40-2k-8)^2-6+k$

$k^2-31k+240=0$

$k=15$ or $k=16$

So, P lies on $y=\mathrm{f}(x)+k$ for $k=15$ or $k=16$.

The claim is agreed.

6. The coordinates of C are $(a,-a-8)$.

Since A lies on the line $x+3y-b=0$, we have

$a+3a-b=0$

$b=4a$

So, the coordinates of B are $(4a,0)$.

Since B is the orthocentre of ΔABC, we have $AB \perp BC$.

$$\left(\frac{0-a}{4a-a}\right)\times\left(\frac{-a-8-0}{a-4a}\right)=-1$$

$a=1$

The perpendicular distance from B to AC $=4(1)-1=3$.

$AC=1-(-1-8)=10$

The area of ΔABC

$=\frac{1}{2}(10)(3)=15$

7. Note that $\angle AOB=90^\circ$.

The circumcentre of ΔOAB lies on AB.

The coordinates of the circumcentre of ΔOAB

$$=\left(\frac{0+20}{2},\frac{-8+0}{2}\right)$$

$=(10,-4)$

8. Let the coordinates of the circumcentre of ΔABC be $G(h,k)$.

Note that $GA=GB=GC$. So, we have

$$\begin{cases}(h-2)^2+(k-5)^2=(h-8)^2+(k-11)^2\\(h-2)^2+(h-5)^2=(h+4)^2+(k-1)^2\end{cases}$$

$$\begin{cases}12h+12k-156=0\\12h+8k-12=0\end{cases}$$

By solving, we have $h=-23, k=36$.

Thus, the coordinates of the circumcentre of ΔABC are $(-23,36)$.

9. Note that AC is horizontal.

The circumcentre of ΔABC lies on the vertical line $x=-4$.

Let the coordinates of the circumcentre of ΔABC be $(-4,k)$.

$(-1+4)^2+(10-k)^2=(5+4)^2+(-2-k)^2$

$24k-24=0$

$k=1$

Thus, the coordinates of the circumcentre of ΔABC are $(-4,1)$

10. The equation of the perpendicular bisector of AB is

$(x-2)^2+(y-3)^2=(x-5)^2+(y-7)^2$

$6x+8y-61=0$

$\begin{cases} 6x+8y-61=0 \\ 5x+y-3=0 \end{cases}$

By solving, we have $x=-\dfrac{37}{34}$, $y=\dfrac{287}{34}$

So, the coordinates of the circumcentre of ΔABC are $\left(-\dfrac{37}{34}, \dfrac{287}{34}\right)$.

The equation of the locus of C is

$\left(x+\dfrac{37}{34}\right)^2+\left(y-\dfrac{287}{34}\right)^2=\left(-\dfrac{37}{34}-2\right)^2+\left(\dfrac{287}{34}-3\right)^2$

$x^2+y^2+\dfrac{37}{17}x-\dfrac{287}{17}y+\dfrac{566}{17}=0$

$17x^2+17y^2+37x-287y+566=0$

11. Since A lies on $y=\dfrac{1}{2}x$, we have $a=2k$.

Since B lies on $y=-\dfrac{4}{5}x$, we have $b=-\dfrac{5}{4}k$.

Note that AB is horizontal.

The circumcentre of ΔOAB lies on the vertical line $x=\dfrac{3}{8}k$.

Let the coordinates of the circumcentre of ΔOAB be $K\left(\dfrac{3}{8}k, c\right)$.

Putting $x=\dfrac{3}{8}k$, $y=c$ in $6x-7y+h=0$, we have

$6\left(\dfrac{3}{8}k\right)-7c+h=0$

$c=\dfrac{9}{28}k+\dfrac{1}{7}h$

Note that $KA=KO$.

$\left(\dfrac{3}{8}k-2k\right)^2+\left(\dfrac{9}{28}k+\dfrac{1}{7}h-k\right)^2=\left(\dfrac{3}{8}k-0\right)^2+\left(\dfrac{9}{28}k+\dfrac{1}{7}h-0\right)^2$

$\dfrac{20}{7}k^2-\dfrac{2}{7}kh=0$

Since $k\neq 0$, we have $10k-h=0$.

Thus, we have $h:k=10:1$.

12. Slope of $AB = (-1)\times(1) = -1$

The equation of AB is

$y+5=-(x+5)$

$y=-x-10$

$\begin{cases} y=-x-10 \\ x-y+10=0 \end{cases}$

Solving, we have $x=-10,\ y=0$.

So, the coordinates of the mid-point of AB are $(-10,0)$.

Thus, the coordinates of B are $(-15,5)$.

Slope of $BC = (-1)\times\left(\frac{1}{3}\right) = -\frac{1}{3}$

The equation of BC is

$y-5=-\frac{1}{3}(x+15)$

$x=-3y$

$\begin{cases} x=-3y \\ 3x-y+10=0 \end{cases}$

Solving, we have $x=-3,\ y=1$.

So, the coordinates of the mid-point of BC are $(-3,1)$.

Thus, the coordinates of C are $(9,-3)$.

Slope of $AC = \frac{-3+5}{9+5} = \frac{1}{7}$

The coordinates of the mid-point of AC are $(2,-4)$.

Therefore, the equation of the perpendicular bisector of AC is

$y+4=-7(x-2)$

$7x+y-10=0$

13. Note that $A(12,0)$ and $B(0,8)$.

Let the coordinates of the circumcentre of ΔABC be $(4,k)$.

$(12-4)^2+(0-k)^2=(0-4)^2+(8-k)^2$

$k=1$

So, the coordinates of the circumcentre of ΔABC are $(4,1)$.

Let the coordinates of C be $\left(c,\dfrac{11c+24}{3}\right)$.

$$(c-4)^2+\left(\frac{11c+24}{3}-1\right)^2=(0-4)^2+(8-1)^2$$

$\dfrac{130}{9}c^2+\dfrac{130}{3}c=0$

$c(c+3)=0$

$c=0\,(\text{rej.})$ or $c=-3$

Thus, the coordinates of C are $(-3,-3)$.