COORDINATE GEOMETRY(III)

Form 6
Vol 6

Part 3B – Intersection of straight line and circle

1. $\begin{cases} y = 3x - 40 \\ x^2 + y^2 - 18x - ky + 50 = 0 \end{cases}$

Putting $y = 3x - 40$ in $x^2 + y^2 - 18x - ky + 50 = 0$, we have

$x^2 + (3x-40)^2 - 18x - k(3x-40) + 50 = 0$

$10x^2 - (3k+258)x + 40k + 1650 = 0$

So, we have

x-coordinate of the mid-point of the chord $AB = \dfrac{3k+258}{2(10)} = \dfrac{3k+258}{20}$

y-coordinate of the mid-point of the chord $AB = 3\left(\dfrac{3k+258}{20}\right) - 40 = \dfrac{9k-26}{20}$

Thus, the coordinates of the mid-point of the chord AB are $\left(\dfrac{3k+258}{20}, \dfrac{9k-26}{20}\right)$.

2. Note that the centre of C lies on the straight line $3x - 4y + 29 = 0$.

Let the coordinates of the centre of C be $\left(h, \dfrac{3h+29}{4}\right)$.

$$(h-0)^2 + \left(\frac{3h+29}{4} - 9\right)^2 = (h-0)^2 + \left(\frac{3h+29}{4} - 1\right)^2$$

$$-\frac{9}{2}(3h+29) + 81 = -\frac{1}{2}(3h+29) + 1$$

$h = -3$

So, the coordinates of the centre of C are $(-3, 5)$.

The radius of C $= \sqrt{(-3)^2 + (5-1)^2} = 5$.

The equation of C is

$(x+3)^2 + (y-5)^2 = 25$

$x^2 + y^2 + 6x - 10y + 9 = 0$

3. B

Note that the mid-point of MN lies on the line $hx-3y=12$.

So, we have $h(6)-3(4)=12$.

$\therefore\ h=4$

Slope of $MN=\dfrac{4}{3}$

The coordinates of the centre of the circle are $\left(7,-\dfrac{k}{4}\right)$.

Note that the line joining the centre to the mid-point of MN is perpendicular to MN.

$$\frac{-\frac{k}{4}-4}{7-6}=-\frac{3}{4}$$

$k=-13$

4. A

Slope of $PQ=\dfrac{3}{4}$

Note that the line joining the centre to the mid-point of PQ is perpendicular to PQ.

Let the coordinates of the mid-point of PQ be $\left(a,\dfrac{3a+20}{4}\right)$.

So, we have

$$\frac{\frac{3a+20}{4}-25}{a+15}=-\frac{4}{3}$$

$a=0$

So, the coordinates of the mid-point of PQ are $(0,5)$.

The perpendicular distance between the centre and the mid-point of PQ

$=\sqrt{(0+15)^2+(5-25)^2}=25$

The radius of $C=\sqrt{25^2+\left(\dfrac{50}{2}\right)^2}=25\sqrt{2}$

The equation of C is

$(x+15)^2+(y-25)^2=1250$

$x^2+y^2+30x-50y-400=0$

5. Note that the equations of the two straight lines are in the form of $y=4x+c$, where c is a constant.

$$\begin{cases} y=4x+c \\ x^2+y^2-6x+2y-7=0 \end{cases}$$

Putting $y=4x+c$ in $x^2+y^2-6x+2y-7=0$, we have

$x^2+(4x+c)^2-6x+2(4x+c)-7=0$

$17x^2+(8c+2)x+c^2+2c-7=0$

Note that $\Delta=0$, we have

$(8c+2)^2-4(17)(c^2+2c-7)=0$

$-4c^2-104c+480=0$

$c=-30$ or $c=4$

Therefore, the equations of the two straight lines are $y=4x-30$ and $y=4x+4$ respectively.

6. Note that the equations of the two tangents are in the form of $y=-x+c$, where c is a constant.

$$\begin{cases} y=-x+c \\ x^2+y^2+10x-4y-3=0 \end{cases}$$

Putting $y=-x+c$ in $x^2+y^2+10x-4y-3=0$, we have

$x^2+(-x+c)^2+10x-4(-x+c)-3=0$

$2x^2+(14-2c)x+c^2-4c-3=0$

Note that $\Delta=0$, we have

$(14-2c)^2-4(2)(c^2-4c-3)=0$

$c^2+6c-55=0$

$c=-11$ or $c=5$

Therefore, the equations of the two tangents are $y=-x-11$ and $y=-x+5$ respectively.

7. The coordinates of the centre of the circle are $(5,-4)$.

Slope of the tangent to the circle at $P=(-1)\times\left(\dfrac{4-5}{0+4}\right)=\dfrac{1}{4}$

The equation of the tangent to the circle at P is

$y=\dfrac{1}{4}(x-4)$

$x-4y-1=0$

8. Let m be the slope of each tangent.

The equations of the two tangents are in the form of:

$$\frac{y+4}{x-3}=m$$

$$y=mx-3m-4$$

$$\begin{cases} y=mx-3m-4 \\ x^2+y^2+2x-8y+9=0 \end{cases}$$

Putting $y=mx-3m-4$ in $x^2+y^2+2x-8y+9=0$, we have

$$x^2+(mx-3m-4)^2+2x-8(mx-3m-4)+9=0$$

$$(1+m^2)x^2-(6m^2+16m-2)x+9m^2+48m+57=0$$

Note that $\Delta=0$, we have

$$(6m^2+16m-2)^2-4(1+m^2)(9m^2+48m+57)=0$$

$$9m^4+48m^3+58m^2-16m+1-(9m^4-48m^3-66m^2-48m-57)=0$$

$$m^2+8m+7=0$$

$$m=-7 \text{ or } m=-1$$

Alternatively,

Note that the shortest distance from the centre to the tangent is the radius.

The coordinates of the centre of the circle are $(-1,4)$.

The radius of the circle $=\sqrt{(-1)^2+(4)^2-9}=2\sqrt{2}$

By the shortest distance formula, we have

$$\left|\frac{m(-1)-(4)-3m-4}{\sqrt{m^2+(-1)^2}}\right|=2\sqrt{2}$$

$$\frac{-4m-8}{\sqrt{m^2+1}}=2\sqrt{2} \text{ or } \frac{-4m-8}{\sqrt{m^2+1}}=-2\sqrt{2}$$

$$-(2m+4)=\sqrt{2m^2+2} \text{ or } 2m+4=\sqrt{2m^2+2}$$

In any cases, we have $m^2+8m+7=0$.

So, we have $m=-7$ or $m=-1$.

Therefore, the equations of the two tangents are $y=-7x+17$ and $y=-x-1$ respectively.

9. (a) The equation of the circle is $x^2 + y^2 = 144$.

(b) Note that $OP \perp PQ$.

Since $\tan\alpha = \frac{3}{4}$, we have $\sin\alpha = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{5}$.

$$OQ = \frac{OP}{\sin\alpha}$$

$$OQ = \frac{12}{\frac{3}{5}} = 20$$

10. (a) (i) Since C touches the y-axis, the coordinates of Q are $(0,5)$.

(ii) Note that the radius of C is 2.

The equation of C is

$$(x+2)^2 + (y-5)^2 = 4$$

$$x^2 + y^2 + 4x - 10y + 25 = 0$$

(b)
$$\begin{cases} y = mx \\ x^2 + y^2 + 4x - 10y + 25 = 0 \end{cases}$$

Putting $y = mx$ in $x^2 + y^2 + 4x - 10y + 25 = 0$, we have

$$x^2 + (mx)^2 + 4x - 10(mx) + 25 = 0$$

$$(1+m^2)x^2 + (4-10m)x + 25 = 0$$

Note that $\Delta = 0$, we have

$$(4-10m)^2 - 4(1+m^2)(25) = 0$$

$$-20m - 21 = 0$$

$$m = -\frac{21}{20}$$

<u>Alternatively,</u>

Let θ be the inclination of the line joining O and P.

$$\tan\theta = -\frac{5}{2}$$

$$\theta = 180° - \tan^{-1}\left(\frac{5}{2}\right)$$

$$\angle QOP = \theta - 90° = 90° - \tan^{-1}\left(\frac{5}{2}\right)$$

Note that $\angle ROP = \angle QOP$.

$$\angle QOR = 2\angle QOP = 180° - 2\tan^{-1}\left(\frac{5}{2}\right)$$

So, we have

$$m = \tan(90° + \angle QOR)$$

$$m = -\frac{21}{20}$$

(c) (i) Note that $\angle PQO = 90°$.

$\angle ORP = 90$ (tangent $\perp$ radius)

So, we have $\angle PQO + \angle ORP = 180°$.

Therefore, O, Q, P and R are concyclic. (opp. $\angle$s supp.)

(ii) Denote the circle that passes through O, Q, P and R by C_1.

Note that the centre of C_1 lies on OP.

The coordinates of the centre of $C_1 = \left(\frac{0-2}{2}, \frac{0+5}{2}\right) = \left(-1, \frac{5}{2}\right)$.

The radius of $C_1 = \sqrt{(0+1)^2 + \left(0 - \frac{5}{2}\right)^2} = \frac{\sqrt{29}}{2}$

The equation of C_1 is

$$(x+1)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{29}{4}$$

$$x^2 + y^2 + 2x - 5y = 0$$

11. (a) Since A lies on L_1, we have

$$(0) = m(-6) + 6\sqrt{2}$$

$$m = \sqrt{2}$$

Since A lies on C_1, we have

$$(-6)^2 + (0)^2 + p(-6) = 0$$

$$p = 6$$

(b) Let the coordinates of B be $(b, \sqrt{2}b + 6\sqrt{2})$.

Putting $x = b$, $y = \sqrt{2}b + 6\sqrt{2}$ in $x^2 + y^2 + 6x = 0$, we have

$$b^2 + (\sqrt{2}b + 6\sqrt{2})^2 + 6b = 0$$

$$b^2 + 10b + 24 = 0$$

$$b = -6 \text{ (rej.)} \quad \text{or} \quad b = -4$$

Thus, the coordinates of B are $(-4, 2\sqrt{2})$.

(c) (i) The coordinates of P are $(0, 6\sqrt{2})$.

(ii) The coordinates of Q are $(-3, 0)$.

Let $x^2 + y^2 + Dx + Ey + F = 0$ be the equation of C_2 where D, E and F are constants.

$$\begin{cases} 36 - 6D + F = 0 & \ldots(1) \\ 72 + 6\sqrt{2}E + F = 0 & \ldots(2) \\ 9 - 3D + F = 0 & \ldots(3) \end{cases}$$

$(1)-(3)$:

$27 - 3D = 0$

$D = 9$

Putting $D = 9$ in (3), we have $F = 18$.

Putting $D = 9$ and $F = 18$ in (2), we have $E = -\dfrac{15\sqrt{2}}{2}$.

The equation of C_2 is

$$x^2 + y^2 + 9x - \frac{15\sqrt{2}}{2}y + 18 = 0$$

$$2x^2 + 2y^2 + 18x - 15\sqrt{2}y + 36 = 0$$

(iii) The coordinates of the centre of C_2 are $\left(-\dfrac{9}{2}, \dfrac{15\sqrt{2}}{4}\right)$.

$$\text{Slope of } L_2 = (-1) \times \left(\frac{0 + \frac{9}{2}}{6\sqrt{2} + \frac{15\sqrt{2}}{4}}\right) = -\sqrt{2}$$

The equation of L_2 is $y = -\sqrt{2}x + 6\sqrt{2}$.

12. (a) Slope of $L = (-1)\times\left(\frac{4-0}{16-0}\right) = -\frac{1}{4}$

The coordinates of the mid-point of $OA = \left(\frac{0+4}{2}, \frac{0+16}{2}\right) = (2,8)$

The equation of L is

$y-8 = -\frac{1}{4}(x-2)$

$x+4y-34=0$

(b) (i) Putting $x=h,\ y=k$ in $x+4y-34=0$, we have

$(h)+4(k)-34=0$

$h=34-4k$

(ii) Note that Γ is a circle with centre M and radius OM.

The equation of the circle is

$(x-h)^2+(y-k)^2=h^2+k^2$

$x^2+y^2-2hx-2ky=0$

$x^2+y^2-2(34-4k)x-2ky=0$

$x^2+y^2+4(2k-17)x-2ky=0$

(c) Note that MA is perpendicular to the tangent.

$\frac{k-16}{34-4k-4}\times\frac{2}{3}=-1$

$k=\frac{29}{5}$

13. (a) Note that

$(61)^2+(108)^2-(3k+39)(61)+(2k-118)(108)-33k-262$

$=3721+11664-183k-2379+216k-12744-33k-262$

$=0$

Thus, C passes through B.

(b) (i) The coordinates of G are $\left(\frac{3k+39}{2}, -k+59\right)$.

Note that A, H and G are collinear.

So, we have

$\frac{-k+59-58}{\frac{3k+39}{2}+109}=\frac{52-58}{-22+109}$

$k=11$

(ii) Note that the equations of the two tangents are in the form of $\frac{y-58}{x+109}=m$ where m is a constant.

$$\begin{cases} y=mx+109m+58 \\ x^2+y^2-72x-96y-625=0 \end{cases}$$

Putting $y=mx+109m+58=0$ in $x^2+y^2-72x-96y-625=0$, we have

$x^2+(mx+109m+58)^2-72x-96(mx+109m+58)-625=0$

$(1+m^2)x^2+(218m^2+20m-72)x+11881m^2+2180m-2829=0$

Note that $\Delta=0$, we have

$(218m^2+20m-72)^2-4(1+m^2)(11881m^2+2180m-2829)=0$

$11881m^4+2180m^3-7748m^2-720m+1296-(11881m^4+2180m^3+9052m^2+2180m-2829)=0$

$-16800m^2-2900m+4125=0$

$m=-\frac{33}{56}$ or $m=\frac{5}{12}$

For $m=-\frac{33}{56}$, we have

$\frac{4225}{3136}x^2-\frac{12675}{1568}x+\frac{38025}{3136}=0$

$x=3$

So, the coordinates of T are $(3,-8)$.

For $m=\frac{5}{12}$, we have

$\frac{169}{144}x^2-\frac{1859}{72}x+\frac{20449}{144}=0$

$x=11$

So, the coordinates of S are $(11,108)$.

(iii) (1) Slope of $BT=\frac{-8-108}{3-61}=2$

Since BS is horizontal, we have $\angle SBT=\tan^{-1}2$.

$\angle SAT=\tan^{-1}\left(\frac{5}{12}\right)+\tan^{-1}\left(\frac{33}{56}\right)$

$\angle SIT=180°-\left(\frac{180°-\angle SAT}{2}\right)$

So, we have $\angle SBT+\angle SIT=180°$.

Thus, B, S, I and T are concyclic (opp. $\angle$s supp.)

Since C passing through B, S and T, I lies on C.

(2) $\angle SGT=360°-2\angle SIT=180°-\angle SAT$

$\angle SAT+\angle SGT=180°$.

Thus, A, S, G and T are concyclic (opp. $\angle$s supp.)

(iv) Since $\angle ASG = 90°$, K lies on AG.

The coordinates of $K = \left(\frac{-109+36}{2}, \frac{58+48}{2}\right) = \left(-\frac{73}{2}, 53\right)$.

The area of $\Delta HSK = \frac{1}{2}(HK)\left(\frac{ST}{2}\right)$

The area of $\Delta ITG = \frac{1}{2}(IG)\left(\frac{ST}{2}\right)$

So, the required ratio is $HK : IG$.

$$HK = \sqrt{\left(-\frac{73}{2}+22\right)^2 + (53-52)^2} = \frac{13\sqrt{5}}{2}$$

$$IG = \sqrt{36^2 + 48^2 + 625} = 65$$

So, we have $\frac{HK}{IG} = \frac{\sqrt{5}}{10}$.

Therefore, the required ratio is $\sqrt{5} : 10$.