COORDINATE GEOMETRY(II)

Form 6
Vol 5

Part 3C – Locus

1. $\begin{cases} 3x+4y-5=0 \\ 12x+5y-7=0 \end{cases}$

By solving, we have $x=\dfrac{1}{11},\ y=\dfrac{13}{11}$.

So, the coordinates of the intersecting point of L_1 and L_2 are $\left(\dfrac{1}{11},\dfrac{13}{11}\right)$.

Note that the required locus is a pair of angle bisectors.
The slopes of the two angle bisectors are:

$$-\tan\left\{\tan^{-1}\left(\frac{12}{5}\right)-\frac{1}{2}\left[\tan^{-1}\left(\frac{12}{5}\right)-\tan^{-1}\left(\frac{3}{4}\right)\right]\right\}=-\frac{9}{7} \text{ and } \frac{7}{9}.$$

So, we have

$$\frac{y-\frac{13}{11}}{x-\frac{1}{11}}=-\frac{9}{7} \quad \text{or} \quad \frac{y-\frac{13}{11}}{x-\frac{1}{11}}=\frac{7}{9}$$

$$9x+7y-\frac{100}{11}=0 \quad \text{or} \quad 7x-9y+10=0$$

Thus, the equations of the required locus are $99x+77y-100=0$ and $7x-9y+10=0$ respectively.

2. The equation of the locus of P is

$$(x-5)^2+(y-1)^2=x^2$$
$$-10x+25+y^2-2y+1=0$$
$$y^2-2y-10x+26=0$$

3. The equation of AB is

$$\frac{y-5}{x+3}=\frac{-7-5}{2+3}$$

$$y=-\frac{12}{5}x-\frac{11}{5}$$

Note that the equations of the locus of P are in the form of $y=-\frac{12}{5}x+c$, where c is a constant.

Since BC is a vertical line with distance 17, we have

$$c=-\frac{11}{5}+17 \quad \text{or} \quad c=-\frac{11}{5}-17$$

$$c=\frac{74}{5} \quad \text{or} \quad c=-\frac{96}{5}$$

Thus, the equation of the locus of P are $y=-\frac{12}{5}x+\frac{74}{5}$ and $y=-\frac{12}{5}x-\frac{96}{5}$ respectively.

4. (a) The locus of P is a circle with centre A and radius AB.

(b) $AB=\sqrt{(0+3)^2+(k-0)^2}=\sqrt{9+k^2}$

$AC=\sqrt{(5+3)^2+(6-0)^2}=10$

So, we have

$$10-\sqrt{9+k^2}=7$$

$$\sqrt{9+k^2}=3$$

$$9+k^2=9$$

$$k=0$$

5. The coordinates of the centre of the circle are $\left(1,-\frac{k}{2}\right)$.

Note that the locus of P passes through the centre of the circle.

So, we have $3(1)-2\left(-\frac{k}{2}\right)-2k=0$.

$\therefore\ k=3$

6. (a) (i) Γ is the angle bisector of $\angle OST$.

(ii) $OS = \sqrt{12^2 + 6^2} = 6\sqrt{5}$

$TS = \sqrt{(-12+15)^2 + 6^2} = 3\sqrt{5}$

Denote the point where Γ intersects the x-axis by N.

The area of $\Delta OSN = \frac{1}{2}(OS)(SN)\sin\angle OSN$

The area of $\Delta TSN = \frac{1}{2}(TS)(SN)\sin\angle TSN$

The ratio of the area of ΔOSN to the area of ΔTSN

$= OS : TS$

$= 2:1$

So, we have $ON : TN = 2:1$.

The coordinates of N

$= \left(\frac{2(-15)}{3}, 0\right)$

$= (-10, 0)$

The equation of Γ is

$\frac{y}{x+10} = \frac{6}{-10+12}$

$3x - y + 30 = 0$

Alternatively,

The slope of $L_1 = \frac{-6}{-12+15} = -2$

The slope of $L_2 = \frac{6}{12} = \frac{1}{2}$

$\therefore$ Slope of $L_1 \times$ Slope of $L_2 = -2 \times \frac{1}{2} = -1$

$\therefore L_1 \perp L_2$

Thus, we have $\angle OST = 90°$.

The slope of Γ

$= \tan\left(\frac{90°}{2} + \tan^{-1}\frac{1}{2}\right)$

$= 3$

The equation of Γ is

$y + 6 = 3(x+12)$

$3x - y + 30 = 0$

(b) Denote the centre of the inscribed circle of ΔABS by $I(h,k)$.

Note that the angle bisector of $\angle ABS$ passes through I.

$\angle ABS = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{2}\right) = 2\tan^{-1}\left(\frac{1}{2}\right)$

$\angle IBS = \tan^{-1}\left(\frac{1}{2}\right)$

$= \text{the inclination of } L_2$

So, the angle bisector of $\angle ABS$ is parallel to the x-axis.

The equation of L_2 is $y = \frac{1}{2}x$.

Putting $y = \frac{1}{2}x$ in $x + 2y = 24$, we have

$x + 2\left(\frac{1}{2}x\right) = 24$

$x = 12$

So, the coordinates of B are $(12, 6)$.

Thus, we have $k = 6$.

Since I lies on Γ, we have

$3h - 6 + 30 = 0$

$h = -8$

Therefore, the coordinates of the centre of the inscribed circle of ΔABS are $(-8, 6)$.

7. (a) The coordinates of G is $(-2,-15)$.

The radius of $C=\sqrt{\left(-\frac{4}{2}\right)^2+\left(-\frac{30}{2}\right)^2+171}=20$

AG

$=\sqrt{(-2+14)^2+(-15-1)^2}$

$=20$,

which equals to the radius of C.

Therefore, A lies on C.

(b) (i) The slope of the line passes through A and G

$=\frac{-15-1}{-2+14}$

$=-\frac{4}{3}$

Let $N(-2,n)$ be the point lying on Γ such that $AN=12$.

$\angle AGN=90°+\tan^{-1}\left(-\frac{4}{3}\right)$

$\approx 36.86989765°$

$n=-15+\frac{12}{\sin\angle AGN}=5$

Thus, we have $N(-2,5)$.

The equation of Γ is

$y-5=-\frac{4}{3}(x+2)$

$y=-\frac{4}{3}x+\frac{7}{3}$

$4x+3y-7=0$

(ii) Note that Φ is the perpendicular bisector of BH.

Denote the mid-point of BH by M.

$$\begin{cases} 4x+3y-7=0 \\ 3x-4y+1=0 \end{cases}$$

By solving, we have $x=1,\ y=1$.

So, the coordinates of M are $(1,1)$.

The area of $\Delta HST = \dfrac{1}{2}(ST)(HM)$

The area of $\Delta KST = \dfrac{1}{2}(ST)(KM)$

Thus, the required ratio is $HM : KM$.

Note that HK is a chord of C.

Denote the mid-point of HK by N.

$$\begin{aligned} NH &= NK \\ &= \sqrt{20^2-12^2} = 16 \end{aligned}$$

$$GM = \sqrt{(1+2)^2+(1+15)^2} = \sqrt{265}$$

$$MN = \sqrt{GM^2-12^2} = 11$$

$$\begin{aligned} HM &= NH - MN \\ &= 16-11 \\ &= 5 \end{aligned}$$

$$\begin{aligned} KM &= NK + MN \\ &= 16+11 \\ &= 27 \end{aligned}$$

Therefore, the required ratio is $5:27$.