EQUATION OF CIRCLE

Form 5

Vol 6

Part 5B – Tangent

1. Let m be the slope of each tangent.

   Then, the equations of the two tangents are in the form of:

   $$\frac{y+2}{x-1}=m$$

   $$mx-y-m-2=0$$

   $$\begin{cases} y=mx-m-2 \\ x^2+y^2-8x-2y+8=0 \end{cases}$$

   Putting $y=mx-m-2$ in $x^2+y^2-8x-2y+8=0$, we have

   $$x^2+(mx-m-2)^2-8x-2(mx-m-2)+8=0$$

   $$(1+m^2)x^2-(2m^2+6m+8)x+m^2+6m+16=0$$

   Note that $\Delta=0$, we have

   $$(2m^2+6m+8)^2-4(1+m^2)(m^2+6m+16)=0$$

   $$m^4+6m^3+17m^2+24m+16-(m^4+6m^3+17m^2+6m+16)=0$$

   $$18m=0$$

   $$m=0$$

Alternatively,

Note that the shortest distance from the centre to the tangent is the radius.

The coordinates of the centre of the circle are $(4,1)$.

The radius of the circle $= \sqrt{4^2 + 1^2 - 8} = 3$

By the shortest distance formula, we have

$$\left| \frac{m(4) - (1) - m - 2}{\sqrt{m^2 + (-1)^2}} \right| = 3$$

$$\frac{3m - 3}{\sqrt{m^2 + 1}} = 3 \quad \text{or} \quad \frac{3m - 3}{\sqrt{m^2 + 1}} = -3$$

$$m - 1 = \sqrt{m^2 + 1} \quad \text{or} \quad -(m - 1) = \sqrt{m^2 + 1}$$

In any cases, we have $-2m = 0$.

So, we have $m = 0$.

Therefore, the equations of the two tangents are $y = -2$ and $x = 1$ respectively.

2. Let m be the slope of each tangent.

The equations of the two tangents are in the form of:

$$\frac{y + 4}{x - 3} = m$$

$$y = mx - 3m - 4$$

$$\begin{cases} y = mx - 3m - 4 \\ x^2 + y^2 + 2x - 8y + 9 = 0 \end{cases}$$

Putting $y = mx - 3m - 4$ in $x^2 + y^2 + 2x - 8y + 9 = 0$, we have

$$x^2 + (mx - 3m - 4)^2 + 2x - 8(mx - 3m - 4) + 9 = 0$$

$$(1 + m^2)x^2 - (6m^2 + 16m - 2)x + 9m^2 + 48m + 57 = 0$$

Note that $\Delta = 0$, we have

$$(6m^2 + 16m - 2)^2 - 4(1 + m^2)(9m^2 + 48m + 57) = 0$$

$$9m^4 + 48m^3 + 58m^2 - 16m + 1 - (9m^4 - 48m^3 - 66m^2 - 48m - 57) = 0$$

$$m^2 + 8m + 7 = 0$$

$$m = -7 \quad \text{or} \quad m = -1$$

Alternatively,

Note that the shortest distance from the centre to the tangent is the radius.

The coordinates of the centre of the circle are $(-1,4)$.

The radius of the circle $= \sqrt{(-1)^2+(4)^2-9} = 2\sqrt{2}$

By the shortest distance formula, we have

$$\left|\frac{m(-1)-(4)-3m-4}{\sqrt{m^2+(-1)^2}}\right| = 2\sqrt{2}$$

$$\frac{-4m-8}{\sqrt{m^2+1}} = 2\sqrt{2} \quad \text{or} \quad \frac{-4m-8}{\sqrt{m^2+1}} = -2\sqrt{2}$$

$$-(2m+4) = \sqrt{2m^2+2} \quad \text{or} \quad 2m+4 = \sqrt{2m^2+2}$$

In any cases, we have $m^2+8m+7=0$.

So, we have $m=-7$ or $m=-1$.

Therefore, the equations of the two tangents are $y=-7x+17$ and $y=-x-1$ respectively.

3. (a) Note that the slope of each tangent is $-\frac{1}{3}$.

The equations of the two tangents are in the form of $y=-\frac{1}{3}x+c$, where c is a constant.

$$\begin{cases} y=-\frac{1}{3}x+c \\ x^2+y^2-6x+2y=0 \end{cases}$$

Putting $y=-\frac{1}{3}x+c$ in $x^2+y^2-6x+2y=0$, we have

$$x^2+\left(-\frac{1}{3}x+c\right)^2-6x+2\left(-\frac{1}{3}x+c\right)=0$$

$$\frac{10}{9}x^2-\left(\frac{2}{3}c+\frac{20}{3}\right)x+c^2+2c=0$$

Note that $\Delta=0$, we have

$$\left(\frac{2}{3}c+\frac{20}{3}\right)^2-4\left(\frac{10}{9}\right)(c^2+2c)=0$$

$$-9c^2+100=0$$

$$c=-\frac{10}{3} \quad \text{or} \quad c=\frac{10}{3}$$

Therefore, the equations of the two tangents are $y=-\frac{1}{3}x-\frac{10}{3}$ and $y=-\frac{1}{3}x+\frac{10}{3}$ respectively.

(b) For $c=-\frac{10}{3}$, we have

$$\frac{10}{9}x^2-\frac{40}{9}x+\frac{40}{9}=0$$

$$x^2-4x+4=0$$

$$(x-2)^2=0$$

$$x=2$$

Thus, the coordinates of the point of contact are $(2,-4)$ for the tangent $y=-\frac{1}{3}x-\frac{10}{3}$.

For $c=\frac{10}{3}$, we have

$$\frac{10}{9}x^2-\frac{80}{9}x+\frac{160}{9}=0$$

$$x^2-8x+16=0$$

$$(x-4)^2=0$$

$$x=4$$

Thus, the coordinates of the point of contact are $(4,2)$ for the tangent $y=-\frac{1}{3}x+\frac{10}{3}$.

4. (a) (i) Since C touches the y-axis, the coordinates of Q are $(0,5)$.

(ii) Note that the radius of C is 2.

The equation of C is

$$(x+2)^2+(y-5)^2=4$$

$$x^2+y^2+4x-10y+25=0$$

(b) $\begin{cases} y = mx \\ x^2 + y^2 + 4x - 10y + 25 = 0 \end{cases}$

Putting $y = mx$ in $x^2 + y^2 + 4x - 10y + 25 = 0$, we have

$x^2 + (mx)^2 + 4x - 10(mx) + 25 = 0$

$(1 + m^2)x^2 + (4 - 10m)x + 25 = 0$

Note that $\Delta = 0$, we have

$(4 - 10m)^2 - 4(1 + m^2)(25) = 0$

$-20m - 21 = 0$

$m = -\dfrac{21}{20}$

Alternatively,

Let θ be the inclination of the line joining O and P.

$\tan\theta = -\dfrac{5}{2}$

$\theta = 180° - \tan^{-1}\left(\dfrac{5}{2}\right)$

$\angle QOP = \theta - 90° = 90° - \tan^{-1}\left(\dfrac{5}{2}\right)$

Note that $\angle ROP = \angle QOP$.

$\angle QOR = 2\angle QOP = 180° - 2\tan^{-1}\left(\dfrac{5}{2}\right)$

So, we have

$m = \tan(90° + \angle QOR)$

$m = -\dfrac{21}{20}$

(c) (i) Note that $\angle PQO = 90°$.

$\angle ORP = 90$ (tangent $\perp$ radius)

So, we have $\angle PQO + \angle ORP = 180°$.

Therefore, O, Q, P and R are concyclic. (opp. $\angle$s supp.)

(ii) Denote the circle that passes through O, Q, P and R by C_1.

Note that the centre of C_1 lies on OP.

The coordinates of the centre of $C_1 = \left(\frac{0-2}{2}, \frac{0+5}{2}\right) = \left(-1, \frac{5}{2}\right)$.

The radius of $C_1 = \sqrt{(0+1)^2 + \left(0-\frac{5}{2}\right)^2} = \frac{\sqrt{29}}{2}$

The equation of C_1 is

$$(x+1)^2 + \left(y-\frac{5}{2}\right)^2 = \frac{29}{4}$$

$$x^2 + y^2 + 2x - 5y = 0$$

5. (a) $C_1 : (x-h)^2 + (y-3)^2 = 3^2$

$x^2 - 2hx + h^2 + y^2 - 6y + 9 = 9$

$x^2 + y^2 - 2hx - 6y + h^2 = 0$

(b) $\begin{cases} x^2 + y^2 - 2hx - 6y + h^2 = 0 \\ y = mx \end{cases}$

$x^2 + (mx)^2 - 2hx - 6(mx) + h^2 = 0$

$(m^2+1)x^2 + (-2h-6m)x + h^2 = 0$

$\Delta = 0$

$(-2h-6m)^2 - 4(m^2+1)(h^2) = 0$

$4h^2 + 24mh + 36m^2 - 4m^2h^2 - 4h^2 = 0$

$24mh + 36m^2 - 4m^2h^2 = 0$

$4m(6h + 9m - mh^2) = 0$

$6h = m(h^2 - 9)$

$m = \frac{6h}{h^2 - 9}$

(c) (i) Sub $D(16,30)$ into L,

$30=16m$

$m=\frac{15}{8}$

$\frac{15}{8}=\frac{6h}{h^2-9}$

$15h^2-135=48h$

$15h^2-48h-135=0$

$h=5$ or $h=-1.8\,(rej.)$

(ii) $\angle POG=\angle ROK=\tan^{-1}\left(\frac{3}{5}\right)$

$=30.964°$

$\angle OPK=\frac{180°-2(30.964°)}{2}$

$=59.036°$

$\angle OPG=\frac{59.036°}{2}=29.518°$

$\angle PGO=180°-29.518°-30.964°$

$=119.52°$

$\approx 120°>90°$

$\therefore \angle$PGO is an obtuse angle

6. (a) Note that $AK \perp L$

Slope of $L = \frac{5}{12}$

Slope of $AK = -\frac{12}{5}$

The equation of AK :

$$\frac{y-5}{x-9} = -\frac{12}{5}$$

$$12x+5y-133=0$$

Let $K(h,k)$

$$12h+5k-133=0$$

$$k = \frac{(133-12h)}{5}$$

The equation of C :

$$(x-h)^2+(y-k)^2=(h-9)^2+(k-5)^2$$

$$x^2-2hx+h^2+y^2-2ky+k^2=h^2-18h+81+k^2-10k+25$$

$$x^2+y^2-2hx-\frac{2(133-12h)}{5}y=-18h-\frac{10(133-12h)}{5}+106$$

$$5x^2+5y^2-10hx+(24h-266)y+800-30h=0$$

(b) Note that $AKBC$ is a square.

Let $B(b,0)$

$$\frac{5-0}{9-b}=\frac{5}{12}$$

$$b=-3$$

$$B(-3,0)$$

Radius of $C = AB$

$$AB=\sqrt{5^2+12^2}=13$$

$$r^2=(h-9)^2+(k-5)^2=13^2$$

$$169h^2-3042h+9464=0$$

$h=14$ or 4

Part 6 – Shortest/Farthest distance

1. Denote the centre of C by G.

Note that the coordinates of G are $(8,5)$.

Let $Q\left(q,-\dfrac{13q+115}{10}\right)$ be the point lying on L such that GQ is perpendicular to L.

Slope of $GQ=(-1)\times\left(-\dfrac{10}{13}\right)=\dfrac{10}{13}$

So, we have

$$\frac{5+\dfrac{13q+115}{10}}{q-8}=-\frac{10}{13}$$

$q=-5$

$\therefore\ Q(-5,-5)$

Note that the distance from P to L attains its minimum as G, P and Q are collinear.

The radius of $C=\sqrt{8^2+5^2+\left(\dfrac{64}{2}\right)}=11$

The distance between G and $Q=\sqrt{(-5-8)^2+(-5-5)^2}=\sqrt{269}$

Therefore, the required distance is $\sqrt{269}-11$.

Alternatively,

The coordinates of the centre of C are $(8,5)$.

The radius of $C=\sqrt{8^2+5^2+\left(\dfrac{64}{2}\right)}=11$

By the shortest distance formula, we have

Shortest distance from the centre of C to L

$$=\left|\frac{13(8)+10(5)+115}{\sqrt{13^2+10^2}}\right|$$

$$=\sqrt{269}$$

Therefore, the required distance is $\sqrt{269}-11$.

2. (a) The locus of P is a circle with centre A and radius AB.

(b) $AB=\sqrt{(0+3)^2+(k-0)^2}=\sqrt{9+k^2}$

$AC=\sqrt{(5+3)^2+(6-0)^2}=10$

So, we have

$10-\sqrt{9+k^2}=7$

$\sqrt{9+k^2}=3$

$9+k^2=9$

$k=0$

Part 7 – Equation of circle + locus

1. (a) (i) The coordinates of B are $(k,2)$.

(ii) Note that Γ is the perpendicular bisector of AB.

Slope of $\Gamma=(-1)\times\left(\frac{k+2}{2-k}\right)=\frac{k+2}{k-2}$

The coordinates of the mid-point of AB are $\left(\frac{k-2}{2},\frac{k+2}{2}\right)$.

The equation of Γ is

$y-\frac{k+2}{2}=\left(\frac{k+2}{k-2}\right)\left(x-\frac{k-2}{2}\right)$

$y=\left(\frac{k+2}{k-2}\right)x$

$(k+2)x-(k-2)y=0$

Note that $(k+2)(0)-(k-2)(0)=0$.

Thus, Γ passes through O.

(iii) Note that AB is a diameter of C.

So, the coordinates of the centre of C are $\left(\frac{k-2}{2},\frac{k+2}{2}\right)$.

The radius of C $=\sqrt{\left(\frac{k-2}{2}-0\right)^2+\left(\frac{k+2}{2}-0\right)^2}=\sqrt{\frac{2k^2+8}{4}}=\sqrt{\frac{k^2+4}{2}}$

The equation of C is

$\left(x-\frac{k-2}{2}\right)^2+\left(y-\frac{k+2}{2}\right)^2=\frac{k^2+4}{2}$

$x^2+y^2-(k-2)x+(k+2)y=0$

(b) (i) Putting $x = -10,\ y = 0$ in C, we have

$$(-10)^2 + (0)^2 - (k-2)(-10) + (k+2)(0) = 0$$

$$k = -8$$

(ii) Note that O is one of the points of intersection of Γ and C.
Denote another point of intersection of Γ and C by D.
Note that $OADB$ is a square inscribed in C.

The diameter of C $= 2\sqrt{\dfrac{(-8)^2 + 4}{2}} = 2\sqrt{34}$

The perimeter of $OADB$

$$= 4\left(\frac{2\sqrt{34}}{\sqrt{2}}\right)$$

$$= 8\sqrt{17}$$

Therefore, the required perimeter is $8\sqrt{17}$.