EQUATION OF CIRCLE

Form 5
Vol 6

Part 2C – Condition of equation of circle

1. Let $x^2+y^2+Dx+Ey+F=0$ be the equation of the circle, where D, E and F are constants.

$$\begin{cases} 4+2E+F=0 & \ldots(1) \\ 36+6D+F=0 & \ldots(2) \\ 164+8D+10E+F=0 & \ldots(3) \end{cases}$$

$(3)-(1)\times 5$:
$144+8D-4F=0 \quad \ldots(4)$
$(4)\times 3-(2)\times 4$:
$288-16F=0$
$F=18$
Putting $F=18$ into (1), we have $E=-11$.
Putting $F=18$ into (2), we have $D=-9$.

Therefore, the equation of the circle is $x^2+y^2-9x-11y+18=0$.

2. Let $x^2+y^2+Dx+Ey+F=0$ be the equation of the circle, where D, E and F are constants.

$$\begin{cases} 26-5D+E+F=0 & \ldots(1) \\ 29+2D+5E+F=0 & \ldots(2) \\ 98+7D-7E+F=0 & \ldots(3) \end{cases}$$

$(2)-(1):$

$3+7D+4E=0 \quad \ldots(4)$

$(3)-(2):$

$69+5D-12E=0 \quad \ldots(5)$

$(4)\times 3+(5):$

$78+26D=0$

$D=-3$

Putting $D=-3$ into (4), we have $E=\dfrac{9}{2}$.

Putting $D=-3$ and $E=\dfrac{9}{2}$ into (1), we have $F=-\dfrac{91}{2}$.

Therefore, the equation of the circle is

$x^2+y^2-3x+\dfrac{9}{2}y-\dfrac{91}{2}=0$

$2x^2+2y^2-6x+9y-91=0$

3. (a) Slope of $PQ = \dfrac{-8-0}{5-1} = -2$

Slope of $QR = \dfrac{-6-(-8)}{9-5} = \dfrac{1}{2}$

$\because$ Slope of $PQ \times$ Slope of $QR = -2 \times \dfrac{1}{2} = -1$

$\therefore\ PQ \perp QR$

Therefore, $\angle PQR$ is a right angle.

(b) Note that the centre of the circumscribed circle lies on PR.

The coordinates of the centre of the circumscribed circle

$= \left(\dfrac{1+9}{2}, \dfrac{0-6}{2}\right)$

$= (5, -3)$

The radius of the circumscribed circle

$= \sqrt{(5-1)^2 + (-3-0)^2}$

$= 5$

The equation of the circumscribed circle is

$(x-5)^2 + (y+3)^2 = 25$

$x^2 + y^2 - 10x + 6y + 9 = 0$

4. (a) The coordinates of P are $(-3,-2)$.

Slope of $BD=(-1)\times\left(\frac{-3+6}{-2-2}\right)=\frac{3}{4}$

Slope of $AD=(-1)\times\left(\frac{-3-1}{-2-1}\right)=-\frac{4}{3}$

Let the coordinates of D be (a,b). Then we have

$$\begin{cases}\frac{b-2}{a+6}=\frac{3}{4}\\ \frac{b-1}{a-1}=-\frac{4}{3}\end{cases}$$

By solving, we have $a=-2, b=5$.

Thus, the coordinates of D are $(-2,5)$.

(b) (i) $\angle PAD=90^\circ$ (tangent $\perp$ radius)

$\angle PBD=90^\circ$ (tangent $\perp$ radius)

So, we have $\angle PAD+\angle PBD=180^\circ$.

Therefore, A, P, B and D are concyclic. (opp. $\angle$s supp.)

(ii) Denote the circle passing through A, P, B and D by C.

Note that centre of C lies on DP.

The coordinates of the centre of C

$$=\left(\frac{-2-3}{2},\frac{5-2}{2}\right)$$

$$=\left(-\frac{5}{2},\frac{3}{2}\right)$$

The radius of C

$$=\sqrt{\left(-3+\frac{5}{2}\right)^2+\left(-2-\frac{3}{2}\right)^2}$$

$$=\frac{5\sqrt{2}}{2}$$

The equation of C is

$$\left(x+\frac{5}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=\frac{25}{2}$$

$$x^2+y^2+5x-3y-4=0$$

Part 3 – Intersection of straight line and circle

1. $\begin{cases} y = 2x - 5 \\ x^2 + y^2 = 10 \end{cases}$

Putting $y = 2x - 5$ in $x^2 + y^2 = 10$, we have

$x^2 + (2x-5)^2 = 10$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x = 1$ or $x = 3$

When $x = 1$, $y = -3$.

When $x = 3$, $y = 1$.

Thus, the two points of intersection are $(1,-3)$ and $(3,1)$.

2. $\begin{cases} y = 4x - 1 \\ x^2 + y^2 + 9x + 4y + 3 = 0 \end{cases}$

Putting $y = 4x - 1$ in $x^2 + y^2 + 9x + 4y + 3 = 0$, we have

$x^2 + (4x-1)^2 + 9x + 4(4x-1) + 3 = 0$

$x^2 + x = 0$

$x(x+1) = 0$

$x = -1$ or $x = 0$

When $x = -1$, $y = -5$.

When $x = 0$, $y = -1$.

Thus, the two points of intersection are $(-1,-5)$ and $(0,-1)$.

3. A

$$\begin{cases} x^2+y^2-10x+16y-2k-49=0 \\ y=\dfrac{4}{3}x-18 \end{cases}$$

Putting $y=\dfrac{4}{3}x-18$ in $x^2+y^2-10x+16y-2k-49=0$, we have

$$x^2+\left(\frac{4}{3}x-18\right)^2-10x+16\left(\frac{4}{3}x-18\right)-2k-49=0$$

$$\frac{25}{9}x^2-\frac{110}{3}x-2k-13=0$$

Note that $\Delta=0$, we have

$$\left(\frac{110}{3}\right)^2-4\left(\frac{25}{9}\right)(-2k-13)=0$$

$$k=-67$$

4. $$\begin{cases} x^2+y^2-6x+2y-7=0 \\ y=6x+k \end{cases}$$

Putting $y=6x+k$ in $x^2+y^2-6x+2y-7=0$, we have

$$x^2+(6x+k)^2-6x+2(6x+k)-7=0$$

$$37x^2+(12k+6)x+k^2+2k-7=0$$

Note that $\Delta>0$, we have

$$(12k+6)^2-4(37)(k^2+2k-7)>0$$

$$k^2+38k-268<0$$

$$(k+19+\sqrt{629})(k+19-\sqrt{629})<0$$

$$-19-\sqrt{629}<k<-19+\sqrt{629}$$

5. (a) The equation of L is $y = mx$.

(b) $\begin{cases} x^2 + y^2 + 6x - 2y + 5 = 0 \\ y = mx \end{cases}$

Putting $y = mx$ in $x^2 + y^2 + 6x - 2y + 5 = 0$, we have

$x^2 + (mx)^2 + 6x - 2(mx) + 5 = 0$

$(1 + m^2)x^2 + (6 - 2m)x + 5 = 0$

Since L is a tangent to the circle, we have

$(6 - 2m)^2 - 4(1 + m^2)(5) = 0$

$2m^2 + 3m - 2 = 0$

$(m + 2)(2m - 1) = 0$

$m = -2$ or $m = \frac{1}{2}$

6. D

$\begin{cases} x^2 + y^2 - 6x + ky + 13 = 0 \\ x = 4 - 2y \end{cases}$

Putting $x = 4 - 2y$ in $x^2 + y^2 - 6x + ky + 13 = 0$, we have

$(4 - 2y)^2 + y^2 - 6(4 - 2y) + ky + 13 = 0$

$5y^2 + (k - 4)y + 5 = 0$

Note that $\Delta \ge 0$, we have

$(k - 4)^2 - 4(5)(5) \ge 0$

$(k + 6)(k - 14) \ge 0$

$k \le -6$ or $k \ge 14$

7. A

$$\begin{cases} 2x^2 + 2y^2 + cx + 6y - c = 0 \\ x = 2y - 1 \end{cases}$$

Putting $x = 2y - 1$ in $2x^2 + 2y^2 + cx + 6y - c = 0$, we have

$$2(2y-1)^2 + 2y^2 + c(2y-1) + 6y - c = 0$$

$$5y^2 + (c-1)y + 1 - c = 0$$

Note that $\Delta < 0$, we have

$$(c-1)^2 - 4(5)(1-c) < 0$$

$$(c+19)(c-1) < 0$$

$$-19 < c < 1$$