COORDINATE GEOMETRY(II)

Form 6
Vol 5

Part 1 – Transformation

1. A

The graph of $y = f(x+3)+4$ is obtained by translating the graph of $y = f(x)$ to the left by 3 units and upwards by 4 units.

2. C

$$\begin{aligned} g(x) &= \log_7 x^8 \\ &= 8\log_7 x \\ &= 8f(x) \end{aligned}$$

The graph of $y = g(x)$ is obtained by enlarging to 8 times of the graph of $y = f(x)$ along the y-axis.

3. D

Denote the graph of $y = \mathrm{f}(x)$ by G.
Note that the vertices of the two graphs are lying in the first quadrant and the fourth quadrant respectively.
So, the transformation of G must involve the reflection with respect to the *x*-axis.
Note that the *x*-intercepts of two graph are different.
So, the transformation of G must involve the translation to the left or right.
Also note that the distances between the *x*-intercepts of each graph are 2 and 4 respectively.
So, the transformation of G must involve the enlargement with respect to the *x*-axis.
Thus, D is the most appropriate choice.

4. D

$$\begin{aligned} f(x) &= 4x^2 + 24nx - 29n^2 \\ &= 4(x^2 + 6nx) - 29n^2 \\ &= 4(x^2 + 6nx + 9n^2 - 9n^2) - 29n^2 \\ &= 4(x+3n)^2 - 65n^2 \end{aligned}$$

So, the coordinates of the vertex of the graph of $y = f(x)$ are $(-3n, -65n^2)$.

Thus, the coordinates of the vertex of the graph of $y = f\left(-\frac{1}{2}x + 4\right)$ are $(6n+8, -65n^2)$.

I is not true.
II is true.
III is true:
Note that the axis of symmetry is a vertical line passing through the vertex.
So, the equation of the axis of symmetry is $x = 6n + 8$, i.e. $x - 6n - 8 = 0$.

5. $$\begin{aligned} y &= f(2-x) \\ &= -\left[-f\left(-(x-1)+1\right)\right] \end{aligned}$$

The graph of $y = f(2-x)$ is obtained by reflecting the graph of $y = -f(x+1)$ along the y-axis, then translating to the right by 1 unit and finally reflecting the resulting graph along the x-axis.

6. (a) The graph of $y = -f(x) - 3$ is obtained by reflecting the graph of $y = f(x)$ along the x-axis and then translating the resulting graph downwards by 3 units.
(b) $(5, 2)$
(c) $2x^2 - 12x + 12$

7. Let $f(x) = 3x^2 - 12x - 9$.
Note that $f(x) = 3(x-2)^2 - 21$.

$$\begin{aligned} y &= -x^2 + 6x + 10 \\ &= -(x-3)^2 + 19 \\ &= -\frac{1}{3}\left\{3\left[(x-1)-2\right]^2 - 21\right\} + 12 \\ &= -\frac{1}{3}f(x-1) + 12 \end{aligned}$$

The graph of $y = -x^2 + 4x + 10$ is obtained by reflecting the graph of $y = f(x)$ along the x-axis, reducing to $\frac{1}{3}$ times of the graph of $y = -f(x)$ along the y-axis, then translating the graph of $y = -\frac{1}{3}f(x)$ to the right by 1 units and finally translating the resulting graph upwards by 12.

8. (a) $f(x)=\frac{1}{k+3}\left[x^2-(2k+2)x+(k+1)^2-(k+1)^2-k+1\right]$

$=\frac{1}{k+3}\left[(x-k-1)^2-k^2-3k\right]$

$=\frac{1}{k+3}(x-k-1)^2-\frac{k(k+3)}{k+3}$

$=\frac{1}{k+3}(x-k-1)^2-k$

Thus, the coordinates of U are $(k+1,-k)$.

(b) Note that the coordinates of V are $(k+1,11+k)$.

i.e. U and V lie on the vertical line $x=k+1$.

So, the orthocentre of ΔOUV lies on the x-axis.

Thus, the coordinates of the orthocentre of ΔOUV are $(-7,0)$.

Note that the line passing through V and the orthocentre of ΔOUV is perpendicular to OU.

Slope of $OU=\frac{-k-0}{k+1-0}=-\frac{k}{k+1}$

So, we have $\frac{11+k-0}{k+7}=\frac{k+1}{k}$.

$\therefore\ k=4$

9. (a) $f(6)=\frac{1}{k+1}\left[(6)^2+3k(6)+9(6)+33k-39\right]$

$=\frac{51k+51}{k+1}$

$=51$

Thus, the graph of $y=f(x)$ passes through S.

(b) (i) $f(x)=\frac{1}{k+1}\left[x^2+(3k+9)x+\left(\frac{3k+9}{2}\right)^2-\left(\frac{3k+9}{2}\right)^2+33k-39\right]$

$=\frac{1}{k+1}\left[\left(x+\frac{3k+9}{2}\right)^2-\frac{9k^2-78k+237}{4}\right]$

$=\frac{1}{k+1}\left(x+\frac{3k+9}{2}\right)^2-\frac{9k^2-78k+237}{4k+4}$

So, the coordinates of U are $\left(-\frac{3k+9}{2},-\frac{9k^2-78k+237}{4k+4}\right)$.

Thus, the coordinates of V are $\left(\frac{3k-3}{2},\frac{9k^2-78k+237}{4k+4}\right)$.

(ii) (1) Note that T is the image of S.

The coordinates of T are $(-12,-51)$.

Note that UT is vertical.

Thus, we have

$$-\frac{3k+9}{2}=-12$$

$$k=5$$

(2) Note that $U(-12,-3)$ and $V(6,3)$.

Let the coordinates of G are $(g,3)$.

Slope of GU × Slope of $VT=-1$

$$\left(\frac{3+3}{g+12}\right)\times\left(\frac{3+51}{6+12}\right)=-1$$

$$g=-30$$

So, the coordinates of G are $(-30,3)$.

$$VT=\sqrt{(6+12)^2+(3+51)^2}=18\sqrt{10}$$

$$GT=\sqrt{(-30+12)^2+(3+51)^2}=18\sqrt{10}$$

Since $VT=GT$, ΔGTV is an isosceles triangle.

Note that the equation of the perpendicular bisector of GV is $x=-12$.

So, K lies on $x=-12$.

Thus, U, K and T are collinear.

(3) Note that $\tan\angle UVT=3$.

$$\angle UVT=\tan^{-1}(3)$$

$$\angle VTK=90°-\tan^{-1}(3)$$

$$\cos\angle VTK=\frac{9\sqrt{10}}{TK}$$

$$TK=30$$

$$\begin{aligned}UK&=UT-TK\\&=(-3+51)-30\\&=18\end{aligned}$$

Note that the required ratio is $TK:UK$.

$$\frac{TK}{UK}=\frac{30}{18}=\frac{5}{3}$$

Thus, the required ratio is $5:3$.

10. (a) $f(x) = x^2 + (10k-12)x + (5k-6)^2 - (5k-6)^2 + 25k^2 - 56k + 33$

$= (x+5k-6)^2 - 25k^2 + 60k - 36 + 25k^2 - 56k + 33$

$= (x+5k-6)^2 + 4k - 3$

Thus, the coordinates of U are $(6-5k, 4k-3)$.

(b) The coordinates of V are $(5k+6, 4k-3)$.

(c) (i) $UV = 5k+6-6+5k = 10k$

$TV = \sqrt{(5k+6-6+k)^2 + (4k-3+3+4k)^2} = \sqrt{100k^2} = 10k$

Since $UV = TV$, ΔUVT is an isosceles triangle.

So, VS is the perpendicular bisector of UT.

Note that K lies on the perpendicular bisector of UT.

Thus, V, K and S are collinear.

(ii) Slope of $TV = \dfrac{4k-3-(-3-4k)}{5k+6-(6-k)} = \dfrac{8k}{6k} = \dfrac{4}{3}$

Note that $\tan\angle UVT = \dfrac{4}{3}$.

$\angle UVT = \tan^{-1}\left(\dfrac{4}{3}\right)$

$\angle UVK = \dfrac{1}{2}\tan^{-1}\left(\dfrac{4}{3}\right)$

Let the coordinates of K be $(6, b)$.

$\tan\angle UVK = \dfrac{4k-3-b}{5k}$

$\dfrac{1}{2} = \dfrac{4k-3-b}{5k}$

$b = \dfrac{3k}{2} - 3$

So, the coordinates of K are $\left(6, \dfrac{3k}{2} - 3\right)$.

The radius of the circumscribed circle $= \sqrt{(5k)^2 + \left(\dfrac{5k}{2}\right)^2} = \dfrac{5\sqrt{5}k}{2}$

The equation of the circumscribed circle is

$(x-6)^2 + \left(y - \dfrac{3k}{2} + 3\right)^2 = \dfrac{125k^2}{4}$

$x^2 + y^2 - 12x - 3ky + 6y - 29k^2 - 9k + 45 = 0$

(iii) Note that the area of ΔUVT is $\frac{1}{2}(10k)^2 \sin \angle UVT$.

So, we have $\frac{1}{2}(10k)^2\left(\frac{4}{5}\right) = 4000$.

$\therefore\ k = 10$

For $k = 10$, we have $K(6,12)$ and $T(-4,-43)$.

Slope of $KQ = \frac{-36-12}{20-6} = -\frac{24}{7}$

Slope of $TQ = \frac{-36+43}{20+4} = \frac{7}{24}$

$\because$ Slope of $KQ \times$ Slope of $TQ = -\frac{24}{7} \times \frac{7}{24} = -1$

$\therefore\ KQ \perp TQ$

$\angle KSU = 90^\circ$ (line joining centre to mid-pt. of chord $\perp$ chord)

So, we have $\angle KSU = \angle KQT = 90^\circ$.

Thus, S, K, Q and T are concyclic. (ext. $\angle =$ int. opp. $\angle$)

The claim is agreed.