TRIGONOMETRY 3D

Form 6
Vol 3

Part 8B – Sliding

1. (a) Let X be the projection of V on the ground.

Note that $\angle VBX = 50°$

Required height

$= VX$

$= 9\sin 50°$

$= 6.8944 \text{cm}$

(b) Area of $\Delta ABC = \frac{1}{2}(15)(\sqrt{10^2 - 7.5^2}) = 49.6078 \text{ cm}^2$

Volume of $VABC = \frac{1}{3}(6.8944)(49.6078) = 114 \text{ cm}^3$

(c) Note that volume of $VABC = \frac{1}{3}(\text{area of } \Delta ABC)(VX) = \frac{1}{3}(\text{area of } \Delta ABC)(VB\sin\angle VBX)$

When $\angle VBX$ decreases to $10°$, $\sin\angle VBX$ decreases, area of ΔABC and VB remain unchanged.

Therefore, the volume decreases.

2. (a) $PR^2 = PQ^2 + QR^2$

$PR^2 = 77^2 + 36^2$

$PR = 85 \text{ cm}$

$QS = SR^2 - QR^2$

$QS^2 = 45^2 - 36^2$

$QS = 27 \text{ cm}$

$PS = PQ - QS$

$PS = 77 - 27 = 50 \text{ cm}$

$PT = PS\cos\angle QPR = (PS)\left(\frac{PQ}{PR}\right)$

$PT = (50)\left(\frac{77}{85}\right)$

$= \frac{770}{17} \text{ cm}$

(b) (i) (1) $RT = PR - PT$

$RT = 85 - \frac{770}{17} = \frac{675}{17}$ cm

$PT^2 = PR^2 + TR^2 - 2(PR)(TR)\cos 32°$

$\left(\frac{770}{17}\right)^2 = PR^2 + \left(\frac{675}{17}\right)^2 - 2(PR)\left(\frac{675}{17}\right)\cos 32°$

$PR^2 - 2\left(\frac{675}{17}\cos 32°\right)(PR) - 475 = 0$

$PR \approx 73.78280979$ cm

Therefore, the required distance is 73.8 cm

(2) Let $s = \frac{PS + PR + SR}{2}$.

The area of ΔPSR

$= \sqrt{s(s - PS)(s - PR)(s - SR)}$

$\approx 1101.295182 \text{ cm}^2$

$ST = PS \sin \angle QPR = (PS)\left(\frac{QR}{PR}\right)$

$ST = (50)\left(\frac{36}{85}\right) = \frac{360}{17}$ cm

The area of ΔPRT

$= \frac{1}{2}(PR)(RT)\sin \angle PRT$

$\approx \frac{1}{2}(73.78280979)\left(\frac{675}{17}\right)\sin 32°$

$\approx 776.2288026 \text{ cm}^2$

Let T' be the projection of T on the horizontal plane.

Note that the required height is TT'.

So, we have

$\frac{1}{3}(\text{area of } \Delta PSR)(TT') = \frac{1}{3}(\text{area of } \Delta PRT)(ST)$

$\frac{1}{3}(1101.295182)(TT') \approx \frac{1}{3}(776.2288026)\left(\frac{360}{17}\right)$

$TT' \approx 14.92586761$ cm

Therefore, the required height is 14.9 cm .

(ii) The volume of the tetrahedron $PSRT$

$= \frac{1}{3}(ST)\left(\frac{1}{2}\right)(PT)(TR)\sin \angle PTR = \frac{1}{6}(ST)(PT)(TR)\sin \angle PTR$

So, the volume of the tetrahedron varies directly as $\sin \angle PTR$.

Note that $\sin \angle PTR$ attains its maximum when $\angle PTR = 90°$.

$PR^2 = PT^2 + RT^2 - 2(PT)(RT)\cos\angle PTR$

$\cos\angle PSR = \dfrac{PS^2 + SR^2 - PR^2}{2(PS)(SR)}$

When $\angle PTR = 90°$, we have

$$PR^2 = \left(\frac{770}{17}\right)^2 + \left(\frac{675}{17}\right)^2$$

Define $\alpha = \cos^{-1}\left(\dfrac{50^2 + 45^2 - \left(\frac{770}{17}\right)^2 + \left(\frac{675}{17}\right)^2}{2(50)(45)}\right)$

Then, we have $\alpha \approx 78.50350675°$

When $\angle PSR$ increases from $35°$ to α, the volume of the tetrahedron increases.

When $\angle PSR$ increases from α to $115°$, the volume of the tetrahedron decreases.

Part 9 – Same/ not same angle? (是否某角)

1. (a) (i) $BX^2 = AB^2 - AX^2$

$BX^2 = (9\sqrt{17})^2 - 9^2$

$BX = 36$ cm

$CX^2 = AC^2 - AX^2$

$CX^2 = 39^2 - 9^2$

$CX = 12\sqrt{10}$ cm

$DX^2 = AD^2 - AX^2$

$DX^2 = 25^2 - 9^2$

$DX = 4\sqrt{34}$ cm

(ii) $\cos\angle BXC = \dfrac{BX^2 + XC^2 - BC^2}{2(BX)(XC)}$

$\cos\angle BXC = \dfrac{36^2 + (12\sqrt{10})^2 - 60^2}{2(36)(12\sqrt{10})}$

$\angle BXC = 108.4349408°$

(b) $\cos\angle DCX = \dfrac{DC^2 + CX^2 - DX^2}{2(DC)(CX)}$

$\cos\angle DCX = \dfrac{56^2 + (12\sqrt{10})^2 - (4\sqrt{34})^2}{2(56)(12\sqrt{10})}$

$\angle DCX \approx 18.43494882°$

$\angle BYC = \angle BXC - \angle DCX$

$\therefore\ \angle BYC = 90°$

$CY = CX \cos \angle DCX$
$CY = 12\sqrt{10} \cos 18.43494882°$
$CY = 36$ cm

$\cos \angle ACD = \dfrac{AC^2 + CD^2 - AD^2}{2(AC)(CD)}$

$\cos \angle ACD = \dfrac{39^2 + 56^2 - 25^2}{2(39)(56)}$

$\angle ACD \approx 22.61986495°$
$AY^2 = AC^2 + CY^2 - 2(AC)(CY)\cos \angle ACD$
$AY^2 \approx 39^2 + 36^2 - 2(39)(36)\cos 22.61986495°$
$AY = 15$ cm
$AY^2 + CY^2 = 15^2 + 36^2 = 1521$
$AC^2 = 39^2 = 1521$
$\because \ AY^2 + CY^2 = AC^2$
$\therefore \ \angle AYC = 90°$
So, we have $AY \perp CD$ and $BY \perp CD$.
Hence the angle between the plane ACD and the plane BCD is $\angle AYB$.
The claim is agreed.

2. (a) $\cos \angle CBD = \dfrac{CB^2 + BD^2 - CD^2}{2(CB)(BD)}$

$\cos \angle CBD = \dfrac{132^2 + 43^2 - 109^2}{2(132)(43)}$

$\angle CBD \approx 49.37067122°$
$DX = BD \sin \angle CBD$
$DX \approx 43 \sin 49.37067122°$
$DX \approx 32.63433774$ cm
$BX^2 = BD^2 - DX^2$
$BX^2 \approx 43^2 - (32.63433774)^2$
$BX = 28$ cm
$AX^2 = AB^2 + BX^2 - 2(AB)(BX)\cos \angle ABC$
$AX^2 = 56^2 + 28^2 - 2(56)(28)\cos 60°$
$AX = 48.49742261$ cm
$AX^2 + BX^2 = 3136$
$AB = 56^2 = 3136$
Since $AX^2 + BX^2 = AB^2$, we have $\angle AXB = 90°$.
So, we have $AX \perp BC$ and $DX \perp BC$.
Hence the angle between the face ABC and the face BCD is $\angle AXD$.
The claim is agreed.

(b) Suppose AX and EF intersect at Y. Note that $\angle AYE = 90°$.

$$AE = \frac{3}{3+4}AB$$

$$AE = \frac{3}{7}(56)$$

$$AE = 24 \text{ cm}$$

$$BE = AB - AE$$
$$BE = 56 - 24$$
$$BE = 32 \text{ cm}$$

Note that $\Delta AEY \sim \Delta ABX$.

$$AY = \frac{3}{7}AX$$

$$AY \approx \frac{3}{7}(48.49742261)$$

$$AY \approx 20.78460969 \text{ cm}$$

$$XY = AX - AY$$
$$XY \approx 27.71281292 \text{ cm}$$

$$\cos\angle AXD = \frac{AX^2 + DX^2 - AD^2}{2(AX)(DX)}$$

$$\cos\angle AXD \approx \frac{48.49742261^2 + 32.63433774^2 - 27^2}{2(48.49742261)(32.63433774)}$$

$$\angle AXD \approx 31.87612336°$$
$$DY^2 = DX^2 + XY^2 - 2(DX)(XY)\cos\angle AXD$$
$$DY^2 \approx (32.63433774)^2 + (27.71281292)^2 - 2(27.71281292)(32.63433774)\cos 31.87612336°$$
$$DY \approx 17.23368794 \text{ cm}$$

$$AY^2 + DY^2 = 729$$

$$AD^2 = 27^2 = 729$$

Since $AY^2 + DY^2 + AD^2$, we have $\angle AYD = 90°$.

Since $DY \perp EF$ and $AY \perp EF$, the plane DEF is perpendicular to the plane ABC.

Hence, the angle between AB and the face DEF is $\angle AEF$.

The claim is agreed.