TRIGONOMETRY 3D

Form 6
Vol 3

Part 7 – Shortest distance

1. Note that volume of $ABCD = \frac{1}{3}$(base area)(height) .

$$\Rightarrow \text{height} = \frac{3\times \text{volume of } ABCD}{\text{base area}}$$

Since the volume of $ABCD$ remains unchanged, the height increases when the base area decreases.
Therefore, D is the most far away from the opposite plane.

2. (a) Note that P is the mid-point of BC, $DP \perp BC$ and $\angle APD = 60°$.
Denote the projection of A on the horizontal ground by A'.

$AA' = AP\sin\angle APD$

$AA' = AP\sin 60°$

$= \frac{\sqrt{3}}{2}AP$

The area of ΔBCD

$= \frac{1}{2}(BC)(DP)$

$= \frac{25}{2}DP$

Note that the volume of the tetrahedron $ABCD = \frac{1}{3}\left(\frac{25}{2}DP\right)\left(\frac{\sqrt{3}}{2}AP\right)$.

$$AP\cdot DP = \frac{2500}{\frac{1}{3}\left(\frac{25}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}$$

$AP\cdot DP = 400\sqrt{3}$

$DP = \frac{400\sqrt{3}}{AP}$

By the cosine formula, we have

$AD^2 = AP^2 + DP^2 - 2(AP)(DP)\cos\angle APD$

$46^2 = AP^2 + \left(\frac{400\sqrt{3}}{AP}\right)^2 - 2(400\sqrt{3})\cos 60°$

$$AP^4-(400\sqrt{3}+2116)AP^2+480000=0$$

$AP^2 \approx 2626.035268$ (rej.) or $AP^2 \approx 182.7850546$

$AP \approx 13.51980231$ cm

≈ 13.5 cm

$DP \approx 51.24485602$ cm

≈ 51.2 cm

(b) $AB^2 = AP^2 + BP^2$

$$AB^2 \approx 182.7850546+\left(\frac{25}{2}\right)^2$$

$AB \approx 18.41290457$ cm

$BD^2 = BP^2 + DP^2$

$$BD^2 \approx \left(\frac{25}{2}\right)^2 + (51.24485602)^2$$

$BD \approx 52.74737215$ cm

Let $s=\dfrac{AB+AD+BD}{2}$.

The area of ΔABD

$=\sqrt{s(s-AB)(s-AD)(s-BD)}$

$\approx 415.5192535\text{ cm}^2$

$\approx 416\text{ cm}^2$

(c) Let the shortest distance from C to the face BCD.

Note that the volume of the tetrahedron $ABCD \approx \frac{1}{3}(415.5192535)d$.

$$d \approx \frac{2500}{\frac{1}{3}(415.5192535)}$$

$d \approx 18.04970513$ cm

Therefore, the shortest distance from C to the face $ABCD$ is 18.0 cm

3. (a) $BE^2 = AE^2 - AB^2$

$BE^2 = 45^2 - 22^2$

$BE \approx 39.25557285$ cm

$CE^2 = CB^2 + BE^2 - 2(CB)(BE)\cos\angle CBE$

$CE^2 \approx 17^2 + (39.25557285)^2 - 2(17)(39.25557285)\cos(145°-90°)$

$CE \approx 32.6259646$ cm

$CE \approx 32.6$ cm

$\sin \angle AEB = \dfrac{AB}{AE}$

$\sin \angle AEB = \dfrac{22}{45}$

$\angle AEB \approx 29.2675776°$

$\cos \angle BEC = \dfrac{BE^2 + EC^2 - BC^2}{2(BE)(EC)}$

$\cos \angle BEC \approx \dfrac{39.25557285^2 + 32.6259646^2 - 17^2}{2(39.25557285)(32.6259646)}$

$\angle BEC \approx 25.26624914°$

$\angle DCE \approx 25.26624914°$

$\angle CDE = 180° - \angle BED = 180° - 90° + \angle ABE \approx 119.2675776°$

$\dfrac{DE}{\sin \angle DCE} = \dfrac{CE}{\sin \angle CDE}$

$\dfrac{DE}{\sin 25.26624914°} \approx \dfrac{32.6259646}{\sin 119.2675776°}$

$DE \approx 15.96337203$ cm

$DE \approx 16.0$ cm

(b) In Figure 3(a), let X be a point lying on BE such that $DX \perp BE$. DX produced meets AE at Y.

$DX = DE \sin \angle DEX$

$DX \approx 15.96337203 \sin(90° - 29.2675776°)$

$DX \approx 13.92558475$ cm

$EX = DE \cos \angle DEX$

$EX \approx 15.96337203 \cos(90° - 29.2675776°)$

$EX \approx 7.804315216$ cm

$XY = EX \tan \angle XEY$

$XY \approx 7.804315216 \tan 29.2675776°$

$XY \approx 4.373772239$ cm

$EY = \dfrac{EX}{\cos \angle XEY}$

$EY \approx \dfrac{7.804315216}{\cos 29.2675776°}$

$EY \approx 8.946352306$ cm

$\cos \angle DEA = \dfrac{DE^2 + EA^2 - AD^2}{2(DE)(EA)}$

$\cos \angle AED \approx \dfrac{15.96337203^2 + 45^2 - 40^2}{2(15.96337203)(45)}$

$\angle AED \approx 61.75863616°$

$DY^2 = DE^2 + EY^2 - 2(DE)(EY)\cos \angle DEY$

$DY^2 \approx 15.96337203^2 + 8.946352306^2 - 2(15.96337203)(8.946352306)\cos 61.75863616°$

$DY \approx 14.1319174$ cm

Note that the inclination of $BCDE$ to the face ABE is $\angle DXY$.

$$\cos\angle DXY = \frac{DX^2 + XY^2 - DY^2}{2(DX)(XY)}$$

$$\cos\angle DXY \approx \frac{13.92558475^2 + 4.373772239^2 - 14.1319174^2}{2(13.92558475)(4.373772239)}$$

$\angle DXY \approx 83.71256092°$

The shortest distance from C to the face ABE

$= DX \sin\angle DXY$

$\approx 13.92558475 \sin 83.71256092°$

≈ 13.8418222 cm

≈ 13.8 cm

4. (a) $$\frac{\sin\angle BAC}{BC} = \frac{\sin\angle ACB}{AB}$$

$$\frac{\sin\angle BAC}{40} = \frac{\sin 60°}{65}$$

$\angle BAC \approx 32.2042275°$ or $\angle BAC \approx 147.7957725°$ (rej.)

$\angle ABC = 180° - \angle ACB - \angle BAC$

$\angle ABC \approx 180° - 60° - 32.2042275°$

$\approx 87.7957725°$

$$\frac{AC}{\sin\angle ABC} = \frac{AB}{\sin\angle ACB}$$

$$\frac{AC}{\sin 87.7957725°} = \frac{65}{\sin 60°}$$

$AC = 75$ cm

$AD^2 = AC^2 - CD^2$

$AD^2 = 75^2 - 32^2$

$AD \approx 67.83067153$ cm

$AD \approx 67.8$ cm

(b) (i) $BD^2 + DC^2 = 24^2 + 32^2 = 1600$

$BC^2 = 40^2 = 1600$

$\because BD^2 + DC^2 = BC^2$

$\therefore \angle BDC = 90°$

Since $CD \perp AD$ and $CD \perp BD$, we have CD is perpendicular to the plane ABD.

Let $s = \dfrac{AB + AD + BD}{2}$.

The area of ΔABD

$= \sqrt{s(s-AB)(s-AD)(s-BD)}$

≈ 778.3957862 cm^2

The volume of the tetrahedron $ABCD$

$\approx \frac{1}{3}(778.3957862)(32)$

$\approx 8302.888386\text{ cm}^3$

$\approx 8300\text{ cm}^3$

(ii) Denote the projection of A on the face BCD by A'. Note that $AA' \perp BD$.

$\cos\angle ADB = \frac{AD^2 + DB^2 - AB^2}{2(AD)(BB)}$

$\cos\angle ADB \approx \frac{67.83067153^2 + 24^2 - 65^2}{2(67.83067153)(24)}$

$\angle ADB \approx 72.9986183°$

$AA' = AD\sin\angle ADB$

$AA' \approx 67.83067153\sin 72.9986183°$

$AA' \approx 64.86631552\text{ cm}$

Let E be the foot of the perpendicular from A to BC such that $AE \perp BC$.

$AE = AB\sin\angle ABC$

$AE \approx 65\sin 87.7957725°$

$AE \approx 64.95190528\text{ cm}$

Note that the inclination of the face ABC to the face BCD is $\angle AEA'$.

$\sin\angle AEA' = \frac{AA'}{AE}$

$\sin\angle AEA' \approx \frac{64.86631552}{64.95190528}$

$\angle AEA' \approx 87.05828697°$

Therefore, the required inclination is $87.1°$.

Part 8A – Sliding

1. (a) $OC = \frac{\sqrt{9^2 - 4.5^2}}{3} \times 2 = \frac{\sqrt{243}}{3}$ cm

Shortest distance $= \sqrt{15^2 - \frac{243}{9}} = \sqrt{198} \approx 14.1\text{ cm}$

(b) Note that $\tan\angle AHO = \frac{OA}{OH}$

Since OA (the height of the pyramid) is a constant, $\tan\angle AHO$ is maximum when OH is minimum. Therefore, $\tan\angle AHO$ is maximum when $AH \perp CD, OH \perp CD$.

Therefore, $\tan\angle AHO$ increases until $AH \perp CD$, then decreases.

2. (a) Let X be a point on AB such that $CX \perp AB, DX \perp AB$

By Heron's formula, area of $\Delta ABC = 20.3332\text{ cm}^2$

$\frac{1}{2}CX(7) = 20.3332$

$CX = 5.8095 \approx 5.81\text{cm}$

$CD = (5.8095)\tan 70° = 15.9614$ cm

(b) Volume $= \frac{1}{3}(15.9614)(20.3332) = 108 \text{ cm}^3$

(c) The volume will decrease since the height of the pyramid decreases when $\angle CXD$ decreases from $70°$ to $30°$.