MORE ABOUT EQUATION

Form 5

Vol 3

Part 1 – Simultaneous Equation (Algebra)

1. $\begin{cases} x^2+y^2 \ +2=4............(1) \\ x-y+2=4............(2) \end{cases}$

Put (2) into (1),

$$x^2+(x-2)^2+2=4$$
$$x^2+x^2-4x+4+2=4$$
$$2x^2-4x+2=0$$
$$x=1$$
$$\therefore y=-1$$

2. $\begin{cases} y=x^2+5x................(1) \\ 4x-y+6=0............(2) \end{cases}$

Put (1) into (2),

$$4x^2-(x^2+5x)+6=0$$
$$x^2+x-6=0$$
$$(x-2)(x+3)=0$$
$$x=2 \text{ or } x=-3$$

When $x=2, y=2^2+5(2)=14$

When $x=-3, y=(-3)^2+5(-3)=-6$

$\therefore x=2, y=14; x=-3, y=-6$

3. $\begin{cases} x^2+y^2 = c^2 \text{..............(1)} \\ x+y=c \text{................(2)} \end{cases}$

Put (2) into (1),

$$x^2+c^2-2cx+x^2-c^2=0$$
$$2x^2-2cx=0$$
$$x=0 \text{ or } x=c$$
$$y=c \text{ or } y=0$$

4. $\begin{cases} y=4x^2+x-5 \text{............(1)} \\ 5x-y-6=0 \text{..............(2)} \end{cases}$

From (2), $y=5x-6$ ……….(3)

Put (1) into (3),

$$4x^2+x-5=5x-6$$
$$4x^2-4x+1=0$$

$$\Delta=(-4)^2-4(4)(1)=0$$

The simultaneous equations have one real solution.

5. (a) $\begin{cases} y=x^2+3x+k \text{............(1)} \\ y=9x-5 \text{...................(2)} \end{cases}$

Put (1) into (2),

$$x^2+3x+k=9x-5$$
$$x^2-6x+(k+5)=0$$

$$\Delta=0$$
$$6^2-4(k+5)=0$$
$$16-4k=0$$
$$k=4$$

(b) $x^2-6x+9=0$

$$x=3$$
$$\therefore y=9(3)-5=22$$

6. C

7. $\begin{cases} y = 2x^2 + mx - 1 & \dots(1) \\ y = 4x - m & \dots(2) \end{cases}$

Put (1) into (2),

$$2x^2 + mx - 1 = 4x + m$$
$$2x^2 + (m-4)x - 1 - m = 0$$

$$\begin{aligned} \Delta &= (m-4)^2 - 4(2)(-1-m) \\ &= m^2 - 8m + 16 + 8 + 8m \\ &= m^2 + 24 \\ &\geq 24 \\ &> 0 \end{aligned}$$

Therefore, the parabola $y = 2x^2 + mx - 1$ and the straight line $y = 4x + m$ intersect at two distinct points for all values of m.

8. $\begin{cases} y = (1-k)x^2 + 3x - k & \dots(1) \\ 3x - y - 1 = 0 & \dots(2) \end{cases}$

From (2), $y = 3x - 1$ $\dots(3)$.
Put (1) into (3),

$$(1-k)x^2 + 3x - k = 3x - 1$$
$$(1-k)x^2 + 1 - k = 0$$

$$\begin{aligned} \Delta &= -4(1-k)(1-k) \\ &= -4(1-k)^2 \end{aligned}$$

For $k \neq 1$, $\Delta < 0$.

Therefore, the parabola $y = (1-k)x^2 - 3x - k$ and the straight line $3x + y + 1 = 0$ do not intersect for all values of k except $k = 1$.

Part 2 – Simultaneous Equation (Graphically)

1. B
2. D
3. (a) Put $x=-3, y=16$

$16=-2(-3)+c$

$c=10$

(b) Put $x=-3, y=16$

$16=2(-3)^2+3(-3)+k$

$k=7$

(c) y - intercept of $P=7$.

(d) $\begin{cases} y=2x^2+3x+7 \\ y=-2x+10 \end{cases}$

$\Rightarrow 2x^2+3x+7=-2x+10$

$2x^2+5x-3=0$

$x=\frac{1}{2}$ or -3 (rej.)

When $x=\frac{1}{2}, y=9$.

$A=\left(\frac{1}{2},9\right)$

Reminder:
***x,y*-intercept** 係數字！不是座標！

4. (a) Equation of L:

$$\frac{y-0}{x-2}=m$$

$$y=mx-2m$$

(b) $\begin{cases} y=x^2-5x+7 \\ y=mx-2m \end{cases}$

$$x^2-(5+m)x+(7+2m)=0$$

$$\Delta=0$$

$$(5+m)^2-4(7+2m)=0$$

$$m^2+2m-3=0$$

$$m=1 \text{ or } -3$$

(c) When $m=1$,

$$x^2-6x+9=0$$

$$x=3$$

$$y=1$$

$$A=(3,1)$$

When $m=-3$,

$$x^2-2x+1=0$$

$$x=1$$

$$y=3$$

$$A=(1,3)$$

Part 3 – $\alpha.\beta$ problems

1. (a) $\begin{cases} y = 2x^2 - 3mx - 1 \\ y = mx + 1 \end{cases}$

$$\Rightarrow \begin{aligned} 2x^2 - 3mx - 1 &= mx + 1 \\ x^2 - 2mx - 1 &= 0 \end{aligned}$$

Since x_1 and x_2 are the roots of $\begin{cases} y = 2x^2 - 3mx - 1 \\ y = mx + 1 \end{cases}$, we have

x_1 and x_2 are the roots of $x^2 - 2mx - 1 = 0$.

Reminder:

Sum of roots $= \frac{-b}{a}$,

Product of root $= \frac{c}{a}$

(b) $x_1 + x_2 = 2m$

$x_1 x_2 = -1$

$$\begin{aligned} (x_2 - x_1)^2 &= (x_1 + x_2)^2 - 4x_1x_2 \\ &= (2m)^2 - 4(-1) \\ &= 4m^2 + 4 \end{aligned}$$

(c) Since the line $y = mx + 1$ passes through A and B, we have

$y_1 = mx_1 + 1$ and $y_2 = mx_2 + 1$.

$\therefore\ y_2 - y_1 = m(x_2 - x_1)$

The distance of AB

$$\begin{aligned} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(x_2 - x_1)^2 + m(x_2 - x_1)^2} \\ &= \sqrt{(m^2 + 1)(x_2 - x_1)^2} \\ &= \sqrt{4(m^2 + 1)^2} \\ &= 2(m^2 + 1) \end{aligned}$$

Thus, we have $2(m^2 + 1) = 20$.

$$\begin{aligned} m^2 + 1 &= 10 \\ m^2 &= 9 \\ m &= 3 \text{ or } -3 \end{aligned}$$

2. (a) $\begin{cases} y = x^2 - x - 3 \\ y = 3x + k \end{cases}$

$x^2 - 4x - (3+k) = 0$

$a_1 + b_1 = 4$

$a_1 b_1 = -3 - k$

(b) x-coordinate of the mid-point $= \dfrac{a_1 + b_1}{2} = 2$

Put $x = 2$ into $y = 3x + k$,

$y = 3(2) + k = 6 + k$

Coordinate of mid-point $= (2, k+6)$

(c) Since $a_1 < 0 < b_1$, P is between A and B.

By section formula,

$P = \left(\dfrac{3a_1 + 2b_1}{5}, \dfrac{3a_2 + 2b_2}{5} \right)$

Since P is the y-intercept of the line, $P = (0, k)$

$\dfrac{3a_1 + 2b_1}{5} = 0$ and $\dfrac{3a_2 + 2b_2}{5} = k$

$\Rightarrow 3a_1 + 2b_1 = 0$ and $\dfrac{3a_2 + 2b_2}{5} = k$

From (a), $a_1 + b_1 = 4$

$\begin{cases} a_1 + b_1 = 4 \\ 3a_1 + 2b_1 = 0 \end{cases}$

By solving, $a_1 = -8, b_1 = 12$

From (a), $a_1 b_1 = -3 - k$

$(-8)(12) = -3 - k$

$k = 93$

3. (a) $\begin{cases} y = x^2 - 6x + 13 \\ y = kx - 3k \end{cases}$

$$x^2 - 6x + 13 = kx - 3k$$
$$x^2 - (k+6)x + 3k + 13 = 0$$

$$\Delta > 0$$
$$(k+6)^2 - 4(3k+13) > 0$$
$$k^2 + 12k + 36 - 12k - 52 > 0$$
$$k^2 - 16 > 0$$
$$k^2 > 16$$

$\therefore\ k < -4$ or $k > 4$

(b) Let α and β be the x-coordinates of P and Q respectively.

Since α and β are the roots of $x^2 - (k+6)x + 3k + 13 = 0$,

we have $\alpha + \beta = k + 6$.

The coordinates of the mid-point of PQ

$$= \left(\frac{\alpha+\beta}{2}, k\left(\frac{\alpha+\beta}{2}\right) - 3k \right)$$

$$= \left(\frac{k+6}{2}, k\left(\frac{k+6}{2}\right) - 3k \right)$$

$$= \left(\frac{k+6}{2}, \frac{k^2}{2} \right)$$

(c) Note that the straight line passes through the mid-point of PQ.

So, we have

$$3\left(\frac{k+6}{2}\right) - \frac{k^2}{2} = 4$$
$$k^2 - 3k - 10 = 0$$
$$(k+2)(k-5) = 0$$

$\therefore\ k = -2$ (rej.) or $k = 5$